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Expansion of a Cylindrical Piston Into a Conducting Fluid

Martin Avery Snyder

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Abstract

The linearized expansion of an infinitely long cylindrical piston into a conducting fluid is considered. The surface of the piston is assumed to be a perfect conductor. The fluid is assumed to be a perfect compressible fluid (free from viscosity and heat conduction) and a perfect electrical conductor, permeated by a uniform magnetic field.

Conical solutions of the linearized Lundquist equations are obtained by an extension of the conical plane wave method first introduced by Gardner. In general, the solution involves both real and complex characteristics. In the course of the analysis, the correct way to handle such coupled problems is elucidated.

We consider two sorts of problems: slow expansions of pistons with roughly circular cross-section, and finite-speed expansions of thin pistons. The linearized problem appears to be a faithful model of the nonlinear problem only in the latter case, i.e., only for pistons whose cross-section is nearly parallel to the undisturbed magnetic field. In other cases, the presence of a magnetic boundary layer is suggested. The linearized solution for a thin piston with finite expansion speed is obtained, and appears to be physically relevant.

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Introduction

In this thesis we consider the linearized expansion of an infinitely long cylindrical piston into a conducting fluid. We assume that the fluid is a perfect compressible fluid (no viscosity or heat conduction) and a perfect electrical conductor, permeated by a constant magnetic field. We obtain conical solutions of the linearized Lundquist equations.

The problem was suggested by two papers of Bernstein, Kulsrud, et al. [2, 7]. They obtained numerical solutions to the three-dimensional problem of an expanding sphere. It was felt that their results were inconclusive, and that an examination of a two-dimensional problem would shed light on the nature of the problem under consideration. We obtain conical solutions by an extension of Gardner's plane wave method [5].

The problem is particularly interesting since there are both real and complex characteristics. Separate representations for the portions of the solution which depend on the real and the complex characteristics are obtained. We first consider slowly expanding bodies. By solving a free-boundary problem we obtain a solution which depends only upon the complex characteristics; the resulting piston cross-section is approximately circular. This solution suggests the presence of a magnetic boundary layer near the piston, since the magnetic field perturbations near the piston are of the same order as the underlying field. For general reasons no solutions involving only the real characteristics can be constructed.

For slow expansions involving pistons of arbitrary cross-section, both the real and complex characteristics must be employed. Using a non-conformal mapping of the $(\frac{x}{t}, \frac{y}{t})$-plane [5] into a complex z-plane, we can reduce the problem to the determination of an analytic function from its boundary values. We present a constructive method for the solution of the boundary value problem.

Finally, we consider the finite-speed expansion of a thin piston which is nearly parallel to the undisturbed magnetic field. In this case we reduce the problem to the same sort of boundary value problem mentioned above. However, in this case, the boundary data is of lower order than the unperturbed flow, so that the resulting solution appears to be physically relevant. For this problem, the linearized problem is probably a faithful model of the nonlinear problem. Treating the piston as a slit, we can write down the solution explicitly in terms of integrals of singular integrals.

1. General Formulation of the Problem

In this first section we shall discuss some of the general features of the problem under consideration, and put on record some general properties of magnetohydrodynamic flows which we shall need later. For the most part, this section is modelled after Gardner's report [5], but a good many of the missing details have been supplied.

The first subsection discusses the linearization of the Lundquist equations, and the reduction to dependence on two space variables. We restrict our discussion to slow expansion velocities of the piston so that linearization is permissible. It does not appear that the linearized solution is always a faithful model of the nonlinear solution. This matter will be discussed in more detail later. The Alfvén wave (given by the components of $\vec{u}$ and $\vec{B}$ along the axis of the piston) is decoupled from the remaining linearized Lundquist equations to obtain the desired two-dimensional formulation.

Subsection two discusses plane waves and the construction of the characteristic conoid. We shall form our solution by superposing conical plane waves. The characteristic conoid will play a central role in our subsequent discussions, since it gives the range of influence of a disturbance at $(x,y,t) = (0,0,0)$ for the linearized Lundquist equations. Parametric equations for the characteristic conoid are obtained.

The third subsection discusses the representation of a conical flow by plane waves. We rely heavily upon Gardner's work [5] in this development. We obtain a formal representation of a conical solution of the linearized Lundquist equations. This representation forms the basis of our subsequent solutions to the piston problem.

Subsection four is devoted to a statement of the well-known jump relations for the Lundquist equations [1]. We shall have occasion to employ these relations in the formulation of appropriate boundary conditions for the problems which we shall consider. They also play a central role in the proof of our uniqueness theorem.

The fifth subsection discusses the boundary conditions which we shall impose at the surface of the piston. Firstly, we assume that the axis of the piston is perpendicular to the constant magnetic field of the undisturbed flow. Secondly, we assume that the tangent to the cross-section of the piston (at fixed time) is perpendicular to the constant undisturbed magnetic field at only a finite number of isolated points. Finally, we assume that $\vec{E} + \vec{u} \times \vec{B} = 0$ on the surface of the piston. The first two assumptions make the formulation of boundary conditions on the surface of the piston particularly simple, and are suggested by Gardner's formulation of the half-plane diffraction problem [5]. The last requirement is equivalent to the assumption that the surface of the piston is a perfect conductor.

The final subsection presents the uniqueness theorem for the piston problem. It is modelled after Gardner's theorem [5], and employs the so-called abc method. Armed with this uniqueness theorem, we can be assured that we are proceeding correctly so long as we can exhibit a solution satisfying whatever (appropriate) conditions we may prescribe.

1-1. Equations of Motion

We assume that the medium in which the piston expands is a perfect compressible fluid (we neglect heat conduction and viscosity) and a perfect electrical conductor. In the undisturbed state the fluid is at rest, and is permeated by a constant magnetic field $\vec{B}_o$. We shall take the axis of the cylindrical piston to be perpendicular to the direction determined by the undisturbed magnetic field, $\vec{B}_o$. The state of motion of such a fluid is specified by the following quantities:

p the scalar pressure
ρ the density
$\vec{u}$ the fluid velocity
$\vec{B}$ the magnetic field
$\vec{E}$ the electric field
$\vec{J}$ the electric current density

For the perfect magnetogas we have described above, these quantities are governed by the Lundquist equations [8], which are as follows:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0 \tag{1-1.1}$$

$$\frac{\partial\vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} + \frac{1}{\rho}\nabla p + \frac{1}{\rho}(\vec{B}\times\vec{J}) = 0 \tag{1-1.2}$$

$$\frac{\partial\vec{B}}{\partial t} + \nabla\times\vec{E} = 0 \tag{1-1.3}$$

$$\nabla\cdot\vec{B} = 0 \tag{1-1.4}$$

$$\nabla\times\vec{B} - \mu_o\vec{J} = 0 \tag{1-1.5}$$

$$\vec{E} + \vec{u}\times\vec{B} = 0 \tag{1-1.6}$$

$$p = p(\rho) \tag{1-1.7}$$

Equation (1-1.1), the continuity equation of ordinary gasdynamics, is a statement of the principle of mass conservation. The next equation, a statement of Newton's second law, is a vector equation known as the ponderomotive equation, the last term being the Lorentz force. Equation (1-1.3) is Faraday's law of induction. Equation (1-1.4) which expresses the absence of magnetic charges, is not independent of equation (1.1-3). Since the divergence of a curl is identically zero, the divergence of equation (1-1.3) is

$$\frac{\partial}{\partial t}(\nabla\cdot\vec{B}) = 0 \ . \tag{1-1.8}$$

Thus, we can regard equation (1-1.4) as an initial condition rather than an independent equation of motion. Ampère's law, neglecting displacement current, is expressed by equation (1-1.5). Equation (1-1.6) is Ohm's law for a perfect conductor. Finally, equation (1-1.7) is the adiabatic equation of state for the fluid. Thus, the Lundquist equations involve the coupling of Maxwell's equations (without displacement current) with the classical equations of gasdynamics. Although the Maxwell equations with displacement current are Lorentz invariant, the Maxwell equations without displacement current are Galilean invariant [3]. Since the gasdynamic equations are likewise Galilean invariant (and we are interested in nonrelativistic motions) it is correct to neglect the displacement current in Ampère's law.

We shall now eliminate $\vec{E}$ and $\vec{J}$ from these equations. First, we solve equation (1-1.5) for $\vec{J}$ obtaining

$$\vec{J} = \frac{1}{\mu_o} (\nabla \times \vec{B}) \tag{1-1.9}$$

and then we substitute this value into equation (1-1.2). Similarly, from equation (1-1.6) we have

$$\vec{E} = \vec{B} \times \vec{u} \tag{1-1.10}$$

and we can use this value in equation (1-1.3). These two substitutions reduce the system of equations (1-1.1) through (1-1.7) to the following system:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \tag{1-1.11}$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} + \frac{1}{\rho}\nabla p + \frac{1}{\rho}(\vec{B}\times[\frac{1}{\mu_o}(\nabla\times\vec{B})]) = 0 \qquad (1\text{-}1.12)$$

$$\frac{\partial \vec{B}}{\partial t} + \nabla \times (\vec{B}\times\vec{u}) = 0 \qquad (1\text{-}1.13)$$

$$\nabla\cdot\vec{B} = 0 \qquad (1\text{-}1.14)$$

$$p = p(\rho) \qquad (1\text{-}1.15)$$

This system of equations is nonlinear. In order to linearize the system, we shall have to restrict our discussion to small disturbances away from the rest state, or, equivalently, to small expansion velocities of the piston. With this restriction in mind, we can linearize the equations (1-1.11) through (1-1.15) by setting

$$\rho = \rho_o(1 + \rho') \qquad (1\text{-}1.16)$$

$$\vec{u} = \vec{u}' \qquad (1\text{-}1.17)$$

$$\vec{B} = B_o(\vec{e}_1 + \vec{B}') \qquad (1\text{-}1.18)$$

where ρ_o and $\vec{B}_o = B_o\vec{e}_1$ are the constant undisturbed values of the fluid density and the magnetic field. ($\vec{e}_1$ is a unit vector which is perpendicular to the axis of the piston.) We now insert (1-1.16), (1-1.17), and (1-1.18) into the nonlinear equations, and neglect products of primed quantities. If we then drop the primes for notational convenience, the resulting linearized Lundquist equations are:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{u} = 0 \qquad (1\text{-}1.19)$$

$$\frac{\partial \vec{u}}{\partial t} + a^2 \nabla \rho + A^2 \vec{e}_1 \times (\nabla \times \vec{B}) = 0 \qquad (1\text{-}1.20)$$

$$\frac{\partial \vec{B}}{\partial t} + \nabla \times (\vec{e}_1 \times \vec{u}) = 0 \qquad (1\text{-}1.21)$$

plus the initial condition

$$\nabla \cdot \vec{B} = 0 \ . \qquad (1\text{-}1.22)$$

In these linearized equations a and A are the undisturbed values of the sound speed and the Alfvén speeds respectively. They are defined to be

$$a^2 = \frac{dp}{d\rho} \qquad (1\text{-}1.23)$$

$$A^2 = \frac{B_o^2}{\mu_o \rho_o} \qquad (1\text{-}1.24)$$

If we now introduce Cartesian coordinates, with the z-axis coincident with the axis of the piston, none of the variables will exhibit a z-dependence. Thus, we can consider the linearized Lundquist equations in two spatial variables, namely x and y. To be definite we shall take the x-direction to be the direction determined by the unit vector $\vec{e}_1$. We shall write $\vec{u} = (u,v,w)$ and $\vec{B} = (B_x, B_y, B_z)$. The equations for the z-components of $\vec{u}$ and $\vec{B}$ are

$$\frac{\partial w}{\partial t} - A^2 \frac{\partial B_z}{\partial x} = 0 \qquad (1\text{-}1.25)$$

$$\frac{\partial B_z}{\partial t} - \frac{\partial w}{\partial x} = 0 \qquad (1\text{-}1.26)$$

These two equations govern the propagation of the Alfvèn wave. It is easy to see that equations (1-1.25) and (1-1.26) reduce to two wave equations, producing waves travelling in the x-direction with speed A. We shall not discuss the Alfven wave further.

The remaining scalar equations of the system (1-1.19) through (1-1.21) depend only on the variables x, y, and t. and are thus the desired two-dimensional formulation. These five scalar equations are as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1\text{-}1.27)$$

$$\frac{\partial u}{\partial t} + a^2 \frac{\partial \rho}{\partial x} = 0 \qquad (1\text{-}1.28)$$

$$\frac{\partial v}{\partial t} + a^2 \frac{\partial \rho}{\partial y} + A^2 \left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x}\right) = 0 \qquad (1\text{-}1.29)$$

$$\frac{\partial B_x}{\partial t} + \frac{\partial v}{\partial y} = 0 \qquad (1\text{-}1.30)$$

$$\frac{\partial B_y}{\partial t} - \frac{\partial v}{\partial x} = 0 \qquad (1\text{-}1.31)$$

The initial condition (1-1.22) becomes

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \qquad (1\text{-}1.32)$$

It is this system of linear equations, depending on two spatial variables, with which we shall concern ourselves.

1-2. Plane Waves

By a plane wave we mean a solution of the equations (1-1.27) through (1-1.32) in which the variables depend upon x, y, and t only in the combination

$$\kappa = Vt - k_x x - k_y k \qquad (1\text{-}2.1)$$

In (1-2.1) V is the propagation speed of the wave, and $\vec{k} = (k_x, k_y)$ is the unit vector in the direction of propagation With the x-direction chosen to coincide with the direction of $\vec{B}_o = B_o \vec{e}_1$, and the z-direction chosen to coincide with the axis of the piston, the y-direction is determined by the requirement that (x,y,z) form an orthogonal triad. We choose $\vec{k}$ to lie in the xy-plane. If we now denote by θ the angle between the two vectors $\vec{k}$ and $\vec{e}_1$ we have

$$k_x = \cos\theta \qquad (1\text{-}2.2)$$

$$k_y = \sin\theta \qquad (1\text{-}2.3)$$

To study the plane wave solutions of the system of equations (1-1.27) through (1-1.32) we make the substitutions $\rho(x,y,t) = \rho(\kappa)$, $u(x,y,t) = u(\kappa)$, $v(x,y,t) = v(\kappa)$, $B_x(x,y,t) = B_x(\kappa)$, and $B_y(x,y,t) = B_y(\kappa)$, where κ is given by (1-2.1). With these substitutions we obtain the following system of equations:

$$V\rho'(\kappa) - k_x u'(\kappa) - k_y v'(\kappa) = 0 \qquad (1\text{-}2.4)$$

$$Vu'(\kappa) - a^2 k_x \rho'(\kappa) = 0 \qquad (1\text{-}2.5)$$

$$Vv'(\kappa) - a^2 k_y \rho'(\kappa) - A^2(k_y B_x' - k_x B_y') = 0 \qquad (1\text{-}2.6)$$

$$VB_x'(\kappa) - k_y v'(\kappa) = 0 \qquad (1\text{-}2.7)$$

$$VB_y'(\kappa) + k_x v'(\kappa) = 0 \qquad (1\text{-}2.8)$$

where the primes denote differentiation with respect to the variable κ. This is a homogeneous system of linear algebraic equations in the derivatives $\rho'(\kappa)$, $u'(\kappa)$, $v'(\kappa)$, $B_x'(\kappa)$, and $B_y'(\kappa)$. In order that this system possess non-trivial solutions the determinant of the coefficients must vanish. Using equations (1-2.2) and (1-2.3), this requirement gives the dependence of the propagation speed V on the angle θ, namely

$$V^4 - (A^2 + a^2)V^2 + A^2a^2\cos^2\theta = 0 \qquad (1\text{-}2.9)$$

Equation (1-2.9) has two positive roots, V_f and V_s, with $V_f \geq V_s$, equality holding only when $A = a$ and $\cos^2\theta = 1$. The corresponding waves are called fast and slow waves respectively.

The dependence of V on the angle θ can be illustrated graphically [4]. The resulting figure is known as the normal speeds locus. Figure 1 is the normal speeds locus for the case $A > a$. The normal speeds locus can easily be constructed for other cases [4].

We shall now consider the totality of plane waves which can propagate in all directions in the xy-plane, and shall suppose that the front of each wave passes through the origin at time $t = 0$. Then, at time $t = t_o$, the various wave fronts form two families of straight lines. The envelope of these straight lines (characteristics) is the $t = t_o$ cross-section of the characteristic conoid. The characteristic conoid gives the range of influence of a disturbance at the origin at time $t = 0$. We can derive the parametric equations for the curves which make up the characteristic conoid in a straightforward manner [4].

Suppose that an initial disturbance is on a plane $X_o(\theta)$ and proceeds in the normal direction given by the angle θ (see figure 2). After a time t the disturbance has moved to a plane $X_t(\theta)$ having traversed a distance $V_n(\theta)t$ where $n = s$ or f, depending on whether the disturbance is a slow or fast wave. We have assumed that the plane $X_o(\theta)$ passes through the origin, and thus can write the equation of this plane as

$$x \cos \theta + y \sin \theta = 0 \ . \tag{1-2.10}$$

The equation of the plane $X_t(\theta)$ is then

$$x \cos \theta + y \sin \theta = \pm V_n(\theta)t \tag{1-2.11}$$

where the plus sign corresponds to the direction of propagation shown in figure 2, while the minus sign corresponds to the opposite direction of propagation.

The envelope of the planes $X_t(\theta)$ is given by the values of (x/t) and (y/t) which satisfy the equation (1-2.11) and its derivative with respect to θ, namely,

$$- x \sin \theta + y \cos \theta = \pm \left(\frac{dV_n(\theta)}{d\theta}\right)t \qquad (1\text{-}2.12)$$

These values are

$$\frac{x}{t} = \pm \left[V_n(\theta)\cos \theta - \left(\frac{dV_n(\theta)}{d\theta}\right) \sin \theta\right] \qquad (1\text{-}2.13)$$

$$\frac{y}{t} = \pm \left[V_n(\theta)\sin \theta + \left(\frac{dV_n(\theta)}{d\theta}\right) \cos \theta\right] \qquad (1\text{-}2.14)$$

We can determine explicit expressions for $V_f(\theta)$ and $V_s(\theta)$ by solving equation (1-2.9). These expressions are

$$V_f(\theta) = A\left[\frac{1}{2}\ (1 + s) + \gamma(\theta)\right]^{\frac{1}{2}} \qquad (1\text{-}2.15)$$

$$V_s(\theta) = A\left[\frac{1}{2}\ (1 + s) - \gamma(\theta)\right]^{\frac{1}{2}} \qquad (1\text{-}2.16)$$

where

$$s = \left(\frac{A}{a}\right)^2 \qquad (1\text{-}2.17)$$

and

$$\gamma(\theta) = \left[(1 + s)^2 - 4s \cos^2\theta\right]^{\frac{1}{2}} \qquad (1\text{-}2.18)$$

Using equations (1-2.13) through (1-2.18) we find that the fast wave envelope is given by the parametric equations

$$\frac{x}{t} = \pm A \cos \theta[\alpha(\theta) - \frac{s \sin^2\theta}{\alpha(\theta)\gamma(\theta)}] \qquad (1\text{-}2.19)$$

$$\frac{y}{t} = \pm A \sin \theta[\alpha(\theta) + \frac{s \sin^2\theta}{\alpha(\theta)\gamma(\theta)}] \qquad (1\text{-}2.20)$$

where

$$\alpha(\theta) = \frac{V_f(\theta)}{A} \qquad (1\text{-}2.21)$$

The slow wave envelope is given by the parametric equations

$$\frac{x}{t} = \pm A \cos \theta[\beta(\theta) + \frac{s \sin^2\theta}{\beta(\theta)\gamma(\theta)}] \qquad (1\text{-}2.22)$$

$$\frac{y}{t} = \pm A \sin \theta[\beta(\theta) - \frac{s \sin^2\theta}{\beta(\theta)\gamma(\theta)}] \qquad (1\text{-}2.23)$$

where

$$\beta(\theta) = \frac{V_s(\theta)}{A} \qquad (1\text{-}2.24)$$

In figure 3 we have sketched the characteristic conoid, and in figure 4 we have sketched a t = constant cross-section of the conoid [4]. The outer branch, ABCD, is the fast wave envelope. The slow wave envelope is composed of the two cusped curves, cFaE and GIKH, and the straight line segment Gc. The distance Oc is given by

$$c = \frac{Aa}{\sqrt{A^2 + a^2}} \qquad (1\text{-}2.25)$$

1-3. Representation of a Conical Field by Plane Waves

We are going to look for a solution of equations (1-1.27) through (1-1.32) which is self-similar, or conical. By this we mean that the variables will depend only on x/t and y/t, or, put another way, the variables will be homogeneous functions of x, y, and t of degree zero. This means that snapshots of the flow at successive instants would be one picture, being enlarged uniformly at a uniform rate. We shall construct such a solution by superposing plane waves that are conical.

The solution to equations (1-1.27) through (1-1.32) can be written in terms of a single function $\psi(x,y,t)$, [5]. In particular, equation (1-1.32) implies the existence of a function $\Phi(x,y,t)$ with

$$B_x = - \Phi_y \tag{1-3.1}$$

$$B_y = + \Phi_x \tag{1-3.2}$$

where the subscripts appended to the function Φ denote partial differentiation. Gardner [5] realized that the choice

$$\Phi(x,y,t) = \psi_{yt}(x,y,t) \tag{1-3.3}$$

enables all the variables to be expressed in terms of simple combinations of third derivatives of the function ψ. Referring back to equations (1-1.27) through (1-1.32) we find that (1-3.1) through (1-3.3) imply that

$$\rho = \frac{1}{a^2}\frac{\partial}{\partial t}[\psi_{tt} - A^2(\psi_{xx} + \psi_{yy})] \qquad (1\text{-}3.4)$$

$$u = -\frac{\partial}{\partial x}[\psi_{tt} - A^2(\psi_{xx} + \psi_{yy})] \qquad (1\text{-}3.5)$$

$$v = -\psi_{ytt} \qquad (1\text{-}3.6)$$

$$B_x = -\psi_{yyt} \qquad (1\text{-}3.7)$$

$$B_y = +\psi_{xyt} \qquad (1\text{-}3.8)$$

Equation (1-1.27) then yields the following equation for the function $\psi(x,y,t)$:

$$\frac{\partial^4\psi}{\partial t^4} - (a^2 + A^2)\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + a^2A^2\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) = 0 \qquad (1\text{-}3.9)$$

Since we are trying to construct a conical flow, we want the third derivatives of ψ to be homogeneous functions of x, y, and t of degree zero. Thus ψ itself must be a homogeneous function of x, y, and t of degree three. Referring to formula (1-2.1) for the form of a plane, we write

$$\psi(x,y,t) = \int_{z_o}^{z} (t - \lambda(s)x - \mu(s)y)^3 h(s)ds \qquad (1\text{-}3.10)$$

We write ψ in the form of an integral so that differentiation will be particularly simple. The point z_o is fixed. The point z is a function of x/t and y/t such that the integrand vanishes at the point z, i.e.,

$$t - \lambda(z)x - \mu(z)y = 0 \tag{1-3.11}$$

Differentiating this relation gives

$$\frac{\partial z}{\partial t} : \frac{\partial z}{\partial x} : \frac{\partial z}{\partial y} = 1 : -\lambda : -\mu \tag{1-3.12}$$

With the aid of these results we can compute the various derivatives of the integral representing the function $\psi(x,y,t)$. The expression (1-3.10) will satisfy the equation (1-3.9) provided that

$$1 - (a^2 + A^2)(\lambda^2 + \mu^2) + a^2A^2\lambda^2(\lambda^2 + \mu^2) = 0 \tag{1-3.13}$$

The significance of λ and μ can be seen by examining (1-3.11) and (1-3.13). First, we see that (1-3.11) is the equation of a straight line in $(\frac{x}{t}, \frac{y}{t})$-space, and that $1/\lambda$ and $1/\mu$ are the intercepts of this line with the x/t and y/t axes respectively. Equation (1-3.13) implies that this straight line is a tangent through the point $P = (\frac{x}{t}, \frac{y}{t})$ to the characteristic conoid of figure 3. In general, through a given point P, there are four such tangents. (Upon eliminating μ from (1-3.13) with the aid of (1-3.11), we obtain a quartic equation for λ.) The nature of the four tangents depends upon the location of the point P, [9,10].

For a point P_1 outside the outer branch of the characteristic conoid the intercepts of these four tangents are real, and the corresponding waves are two fast waves and two slow waves. For a point P_2 between the inner and outer branches of the characteristic conoid, two of the tangents have real intercepts, and the corresponding waves are slow waves. The other two tangents from such a point have complex intercepts. These two cases are illustrated in figure 5. Finally, for a point P_3 inside the inner branch of the characteristic conoid, the intercepts of all four tangents are real, and the resulting waves are four slow waves. This situation is shown in figure 6. We shall be concerned with the region between the inner and outer branches of the characteristic conoid. We shall refer to the solution corresponding to the two tangents with real intercepts as the hyperbolic wave, and the solution corresponding to the two tangents with complex intercepts will be called the elliptic solution.

In accordance with (1-3.10) we shall write our solution (formally) in the form

$$\rho = \int_{z_o}^{z} P(s)ds \tag{1-3.14}$$

$$u = \int_{z_o}^{z} U(s)ds \tag{1-3.15}$$

$$v = \int_{z_o}^{z} V(s)ds \qquad (1\text{-}3.16)$$

$$B_x = \int_{z_o}^{z} B_1(s)ds \qquad (1\text{-}3.17)$$

$$B_y = \int_{z_o}^{z} B_2(s)ds \qquad (1\text{-}3.18)$$

In the hyperbolic regime, z will be real, and these integrals will be real. In the elliptic solution, z will be complex and these integrals will be complex contour integrals of analytic functions. We shall impose further conditions on the integrands, the paths of integration, and choose the point z_o. We can differentiate these formal expressions, and use equations (1-1.27) through (1-1.32) to obtain relations between P, U, B_1, B_2, and V (V is singled out for convenience only.) If we use the relations contained in (1-3.12) we find that

$$P(s) = \frac{1}{a^2\mu(s)} [1 - A^2(\lambda^2(s) + \mu^2(s))]V(s) \qquad (1\text{-}3.19)$$

$$U(s) = \frac{\lambda(s)}{\mu(s)} [1 - A^2(\lambda^2(s) + \mu^2(s))]V(s) \qquad (1\text{-}3.20)$$

$$B_1(s) = \mu(s)V(s) \qquad (1\text{-}3.21)$$

$$B_2(s) = - \lambda(s)V(s) \qquad (1\text{-}3.22)$$

1-4. Jump Relations

For future reference we shall discuss in this subsection the propagation of discontinuities in a solution of the equations (1-1.27) through (1-1.32). Let us suppose that we have a solution of these equations which tends to different limits as the point (x,y,t) tends to the two different sides of a surface S in (x,y,t) space, so that the quantities ρ, u, v, B_x, and B_y have jump discontinuities In such circumstances the differential equations (1-1.27) through (1-1.32) break down; in their place we have the jump relations given below. These jump relations can be obtained in a variety of ways, for instance, by considering weak solutions of the equations under discussion. We denote the components of the normal vector to S by n_x, n_y, and n_t. The desired jump relations are as follows [1]:

$$n_t[\rho] + n_x[u] + n_y[v] = 0 \tag{1-4.1}$$

$$n_t[u] + a^2 n_x[\rho] = 0 \tag{1-4.2}$$

$$n_t[v] + a^2 n_y[\rho] + A^2(n_y[B_x] - n_x[B_y]) = 0 \tag{1-4.3}$$

$$n_t[B_x] + n_y[v] = 0 \tag{1-4.4}$$

$$n_t[B_y] - n_x[v] = 0 \tag{1-4.5}$$

$$n_x[B_x] + n_y[B_y] = 0 \tag{1-4.6}$$

where [Q] denotes the jump in the quantity Q across the surface S. We can eliminate the jumps in the five variables from these six equations, obtaining an equation relating the components of the normal vector to the surface S. This equation is

$$n_t^4 - (A^2 + a^2)n_t^2(n_x^2 + n_y^2) + A^2a^2n_x^2(n_x^2 + n_y^2) = 0 \qquad (1\text{-}4.7)$$

Equation (1-4.7) implies that the surface S is a characteristic surface of the system of equations (1-1.27) through (1-1.32).

1-5. Boundary Conditions on the Piston

In this subsection we shall discuss the form of the linearized boundary conditions on the piston. We shall use a slightly different form of the boundary conditions than those obtained here when we discuss in the last section pistons which are nearly parallel to the undisturbed magnetic field $\vec{B}_o$. All vectors in this subsection are two-dimensional vectors in $(\frac{x}{t}, \frac{y}{t})$-space.

We shall impose only one restriction on the shape of the piston, namely that $(\vec{n}\cdot\vec{e}_1) = 0$ at only a finite number of isolated points. Here $\vec{n}$ is the unit normal to the surface of the piston, and $\vec{e}_1$ is a unit vector in the $\frac{x}{t}$ direction.

This assumption permits us to cast the boundary conditions into a useful form. We shall denote by $\vec{U}_p$ the velocity with which the piston is expanding.

The first boundary condition to be applied at the surface of the piston is that no fluid flows through that surface, i.e.,

$$\vec{u}\cdot\vec{n} = \vec{U}_p\cdot\vec{n} \tag{1-5.1}$$

We shall denote the magnetic field on the surface of the piston by $\vec{B}_p$, and the corresponding electric field by $\vec{E}_p$. We assume that on the surface of the piston

$$\vec{E}_p + \vec{U}_p \times \vec{B}_p = 0 \tag{1-5.2}$$

This assumption is equivalent to the requirement that the surface of the piston is a perfect conductor. Using Ohm's law for the perfectly conducting fluid, (1-1.6), we can express the condition that the tangential component of the electric field is continuous across the surface of the piston by writing

$$\vec{n} \times (\vec{B} \times \vec{u}) = \vec{n} \times (\vec{B}_p \times \vec{U}_p) \tag{1-5.3}$$

We shall now linearize equation (1-5.3). In analogy with (1-1.18) we write

$$\vec{B}_p = B_o(\vec{e}_1 + \vec{B}_p') \tag{1-5.4}$$

If we denote by $d\vec{S}$ the element of surface area on the piston, we know that

$$\frac{D}{Dt}\int_{S(t)} \vec{B}\cdot d\vec{S} = \int \left(\frac{\partial \vec{B}}{\partial t} + \nabla \times (\vec{B} \times \vec{u})\right)\cdot d\vec{S} \qquad (1\text{-}5.5)$$

The nonlinear equation (1-1.13) implies that the integrand in the right hand integral is zero, so that on any piece of the piston, $S(t)$, we must have

$$\int_{S(t)} \vec{B}\cdot d\vec{S} = \text{constant} \qquad (1\text{-}5.6)$$

For a conical flow, $\vec{B}$ remains constant while $S(t)$ changes with time. Thus, the constant in (1-5.6) must be zero. This can be achieved by taking $\vec{B}$ to be tangential to the surface of the piston. Thus, one might suspect that the linearization (1-5.4) would be justified only for a piston whose surface is nearly parallel to the undisturbed magnetic field. This conclusion is borne out by our subsequent analysis. It should be pointed out, however, that the *a priori* conclusion is based upon the nonlinear equations; this conclusion cannot be drawn from the linearized system. *A priori*, the linearized equations and boundary conditions are consistent.

Using equations (1-1.18) and (1-5.4) in (1-5.3) we obtain the linearized boundary condition

$$\vec{n} \times (\vec{e}_1 \times \vec{u}) = \vec{n} \times (\vec{e}_1 \times \vec{U}_p) \tag{1-5.7}$$

Upon expanding the vector triple products we find that

$$(\vec{n}\cdot\vec{u})\vec{e}_1 - (\vec{n}\cdot\vec{e}_1)\vec{u} = (\vec{n}\cdot\vec{U}_p)\vec{e}_1 - (\vec{n}\cdot\vec{e}_1)\vec{U}_p \tag{1-5.8}$$

We can rearrange this equation to obtain

$$(\vec{n}\cdot\vec{e}_1)(\vec{u} - \vec{U}_p) = 0 \tag{1-5.9}$$

With our assumption on the vanishing of $(\vec{n}\cdot e_1)$, and the desired continuity of $\vec{u}$, we conclude that

$$\vec{u} = \vec{U}_p \tag{1-5.10}$$

on the surface of the piston. Thus, the requirements that no fluid flow through the surface of the piston, and that the tangential component of the electric field is continuous across that surface lead to the two boundary conditions

$$u = \vec{U}_p\cdot\vec{e}_1 = U_1 \tag{1-5.11}$$

$$v = \vec{U}_p\cdot\vec{e}_2 = U_2 \tag{1-5.12}$$

at the surface of the piston. ($\vec{e}_2$ is a unit vector in the y-direction.)

Finally, we must require that the normal component of the magnetic field is continuous across the surface of the piston. In terms of the perturbations of the magnetic field on the piston and in the fluid we require that

$$\vec{B}\cdot\vec{n} = \vec{B}_p\cdot\vec{n} \tag{1-5.13}$$

This relation provides the definition of $\vec{B}_p\cdot\vec{n}$ on the surface of the piston. We can then solve an interior problem to obtain the magnetic field throughout the piston.

In the last section, when we consider pistons which are everywhere nearly parallel to the undisturbed magnetic field, we shall consider a slightly different form of these boundary conditions.

1-6. Uniqueness of the Solution

We shall model our uniqueness theorem upon that given by Gardner [5]. We suppose that ρ, u, v, B_x, and B_y are differentiable in the open region R of (x,y,t)-space which is outside the piston and does not include certain surfaces, and that these quantities satisfy the equations (1-1.27) through (1-1.32) in R. The surfaces include the surface of the piston, the characteristic surfaces ABCD, cFaE, GIKH, and that portion of the line segment Gc which lies outside of the piston, and certain line segments across which the variables are continuous, but their various first derivatives are not (see figure 7). We assume that at a characteristic surface the jump relations (1-4.1) through (1-4.6) hold. Finally, we assume that

$$\rho = u = v = B_x = B_y = 0 \tag{1-6.1}$$

for $t > t_o$ outside some bounded region of the xy-plane. This is a statement of the fact that magnetohydrodynamic waves propagate with finite velocity.

To show that such a solution is unique, we suppose that there could be two solutions, ρ_1, u_1, v_1, B_{x1}, B_{y1} and ρ_2, u_2, v_2, B_{x2}, B_{y2}. We shall consider the difference of these two solutions: $\rho = \rho_1 - \rho_2$, $u = u_1 - u_2$, $v = v_1 - v_2$, $B_x = B_{x1} - B_{x2}$, and $B_y = B_{y1} - B_{y2}$. We shall now show that this solution is identically zero.

The proof employs the abc or energy method. For this method to succeed, the various variables must be square integrable. We multiply (1-1.27) by $a^2\rho$, (1-1.28) by u, (1-1.29) by v, (1-1.30) by A^2B_x and (1-1.31) by A^2B_y. We then add the six resulting equations to obtain a relation which may be written in the form

$$\frac{1}{2}\frac{\partial}{\partial t}(a^2\rho^2 + u^2 + v^2 + A^2B_x^2 + A^2B_y^2) + \frac{\partial}{\partial x}(a^2\rho u - A^2vB_y)$$
$$+ \frac{\partial}{\partial y}(a^2\rho v + A^2vB_x) = 0 \tag{1-6.2}$$

We now integrate (1-6.2) over all (x,y,t) space, exclusive of the piston, with $t < t_o$. On the surface of the piston, the solution will satisfy the homogeneous boundary conditions

$$u = v = \vec{B}\cdot\vec{n} = 0 \qquad (1\text{-}6.3)$$

This, combined with the square integrability of the solution, insures that the boundary terms arising from the integration at the surface of the piston will vanish.

Moreover, the boundary terms arising at a characteristic surface S likewise contribute nothing. If we denote the values of the variables on one side of S by ρ, u, v, B_x, and B_y, and the values on the other side of S by $\bar{\rho}$, $\bar{u}$, $\bar{v}$, $\bar{B}_x$, and $\bar{B}_y$, then the integrand of the surface integral arising at S is

$$\frac{1}{2} n_t(a^2\bar{\rho}^2 + \bar{u}^2 + \bar{v}^2 + A^2\bar{B}_x^2 + A^2\bar{B}_y^2) + n_x(a^2\bar{\rho}\bar{u} - A^2\bar{v}\bar{B}_y)$$

$$+ n_y(a^2\bar{\rho}\bar{v} + A^2\bar{v}\bar{B}_x) - \frac{1}{2} n_t(a^2\rho^2 + u^2 + v^2 + A^2B_x^2 + A^2B_y^2)$$

$$- n_x(a^2\rho u - A^2vB_y) - n_y(a^2\rho v + A^2vB_x) \qquad (1\text{-}6.4)$$

We can regroup equation (1-6.4) to obtain the expression

$$\begin{aligned}
&\tfrac{1}{2}(\rho+\overline{\rho})\left\{n_t[\rho]+n_x[u]+n_y[v]\right\}+\tfrac{1}{2}(u+\overline{u})\left\{n_t[u]+a^2n_x[\rho]\right\} \\
&+\tfrac{1}{2}(v+\overline{v})\left\{n_t[v]+a^2n_y[\rho]+A^2n_y[B_x]-A^2n_x[B_y]\right\} \qquad (1\text{-}6.5) \\
&+\tfrac{1}{2}(B_x+\overline{B}_x)\left\{n_t[B_x]+n_y[v]\right\}+\tfrac{1}{2}(B_y+\overline{B}_y)\left\{n_t[B_y]-n_x[v]\right\}
\end{aligned}$$

Referring to the jump relations (1-4.1) through (1-4.6) we see that (1-6.5) is identically zero.

Finally, the integrated terms arising at a line where the solution is continuous and its derivatives are not also vanish. (Such a line is also a characteristic, but not part of the characteristic conoid.) Therefore, upon integrating (1-6.2) we are left with the following integral:

$$\frac{1}{2}\iint\limits_{t=t_o}(a^2\rho + u^2 + v^2 + A^2B_x^2 + A^2B_y^2)dxdy = 0 \qquad (1\text{-}6.6)$$

Equation (1-6.6) implies that

$$\rho = u = v = B_x = B_y = 0 \qquad (1\text{-}6.7)$$

which is what we wanted to show.

2. The Elliptic Solution

In the first section we discussed the nature of the tangents to a t = constant section of the characteristic conoid. In particular, for a point P between the slow and fast wave envelopes, we saw that two of the four tangents to the characteristic conoid had complex $\frac{x}{t}$ and $\frac{y}{t}$-intercepts. It is these characteristics that are employed in the construction of the elliptic solution.

In the first subsection we obtain an explicit representation of the intercepts of these tangents, $\frac{1}{\lambda}$ and $\frac{1}{\mu}$, in terms of a complex parameter z. The relation between z and $\frac{x}{t}$ and $\frac{y}{t}$ provides a non-conformal mapping of the $(\frac{x}{t}, \frac{y}{t})$-plane into the complex z-plane. Following Gardner [5], we devote the next subsection to a detailed examination of this mapping. By employing this mapping, we formulate our problem as a boundary value problem for an analytic function in the z-plane.

In the final subsection of this chapter we discuss this formulation of the problem, and its solution. The solution is completely determined up to one analytic function. Certain general restrictions are placed on this function, but its complete determination depends upon the boundary conditions imposed on the surface of the piston, and upon the shape of the piston cross-section.

2-1. The Form of λ and μ

In the first section we wrote our solution in terms of various third derivatives of a homogeneous function of x, y, and t of degree three, namely

$$\psi(x,y,t) = \int_{z_0}^{z} (t - \lambda(s)x - \mu(s)y)^3 h(s)ds \qquad (2\text{-}1.1)$$

The formal solution was then given by equations (1-3.4) through (1-3.8). We concluded at that point that λ and μ must also satisfy the two relations

$$t - \lambda(z)x - \mu(z)y = 0 \qquad (2\text{-}1.2)$$

$$1 - (a^2 + A^2)(\lambda^2(z) + \mu^2(z)) + a^2A^2\lambda^2(z)(\lambda^2(z) + \mu^2(z)) = 0 \qquad (2\text{-}1.3)$$

If we insert a specific form for $\mu^2(z)$ into (2-1.3) we obtain a quadratic equation for $\lambda^2(z)$. If, on the other hand, we substitute a specific form for $\lambda^2(z)$ into (2-1.3) we obtain a linear equation for $\mu^2(z)$. We follow this latter course of action and write [5]

$$\lambda(z) = \frac{1}{z} \qquad (2\text{-}1.4)$$

Geometrically, z is the complex $(\frac{x}{t})$-intercept of a tangent to the characteristic conoid. Using (2-1.4) in (2-1.3) we obtain

$$\mu(z) = \left(\frac{c}{aA}\right)\frac{1}{z}\sqrt{\frac{(z^2 - A^2)(z^2 - a^2)}{z^2 - c^2}} \qquad (2\text{-}1.5)$$

In order that $\mu(z)$ be a single-valued function of the complex variable z, we must make certain cuts in the z-plane. We do this by making four cuts on the real axis, from -A to -∞, from -c to -a, from c to a, and from A to ∞ (see figure 8). To choose a definite branch of the square root in (2-1.5) we write

$$\mu(z) = \frac{i}{z}\sqrt{\frac{(1 - z^2/A^2)(1 - z^2/a^2)}{1 - z^2/c^2}} \qquad (2\text{-}1.6)$$

and choose this square root to be +1 for z = 0.

The solutions which result from consideration of the two complex characteristics are complex themselves. We want to form a combination of these two solutions which is real. Since the two complex solutions are complex conjugates of one another, we shall consider half their difference, or equivalently, the imaginary part of either solution. We can accomplish this by modifying formulae (2-1.1). We take as our definition of $\psi(x,y,t)$ for the elliptic solution the formula

$$\psi(x,y,t) = \operatorname{Im}\int_{z_o}^{z}(t - \lambda(s)x - \mu(s)y)^3 h(s)ds \qquad (2\text{-}1.7)$$

In this formula we shall take h(s) to be an analytic function. The integral will be interpreted as a contour integral along an appropriately specified contour.

Equation (2-1.2) gives the relation between z and $\frac{x}{t}$ and $\frac{y}{t}$, namely

$$\left(\frac{x}{t}\right) + \left(\frac{c}{aA}\right)\sqrt{\frac{(z^2 - A^2)(z^2 - a^2)}{z^2 - c^2}}\left(\frac{y}{t}\right) = z \tag{2-1.8}$$

By considering the real and imaginary parts of this equation we find that

$$\frac{y}{t} = \left(\frac{aA}{c}\right)\left\{\frac{\operatorname{Im}(z)}{\operatorname{Im}\sqrt{\frac{(z^2 - A^2)(z^2 - a^2)}{z^2 - c^2}}}\right\} \tag{2-1.9}$$

$$\frac{x}{t} = \operatorname{Re}(z) - \left(\frac{c}{aA}\right)\left\{\operatorname{Re}\sqrt{\frac{(z^2 - A^2)(z^2 - a^2)}{z^2 - c^2}}\right\}\left(\frac{y}{t}\right) \tag{2-1.10}$$

Equation (2-1.8) provides a non-conformal mapping of the $(\frac{x}{t}, \frac{y}{t})$-plane into the complex z-plane. We shall consider this mapping in depth in the next subsection.

2-2. The Mapping $(\frac{x}{t}) + (\frac{c}{aA})\sqrt{\frac{(z^2 - A^2)(z^2 - a^2)}{z^2 - c^2}}\,(\frac{y}{t}) = z$

As we pointed out in the last subsection, this is a non-conformal mapping of the $(\frac{x}{t}, \frac{y}{t})$-plane into the z-plane. This mapping possesses some very special properties which will aid greatly in the construction of solutions to our problem. In this subsection we shall prove that this mapping possesses the following five properties:

(i) The mapping (2-1.8) of the slit z-plane of figure 8 is one-one and differentiable into a region R of the $(\frac{x}{t}, \frac{y}{t})$-plane.

(ii) Points in R for which $\frac{y}{t} = 0$ are invariant under this mapping.

(iii) Near the origin, the mapping is approximately the identity mapping; more precisely

$$z = (\frac{x}{t}) + i(\frac{y}{t})[1 + O(|z|^4] \qquad (2\text{-}2.1)$$

for small $|z|$

(iv) The boundary curves of R determine characteristic surfaces.

(v) A point on the boundary of R is mapped into the point P on the $\frac{x}{t}$-axis where the tangent from that point on the boundary of R intersects the $\frac{x}{t}$-axis.

Proof of (i): We take

$$z = \xi + i\eta \tag{2-2.2}$$

and write

$$\lambda(z) = \lambda_r(\xi,\eta) + i\lambda_i(\xi,\eta) \tag{2-2.3}$$

$$\mu(z) = \mu_r(\xi,\eta) + i\mu_i(\xi,\eta) \tag{2-2.4}$$

Recalling the representations (2-1.4) and (2-1.5) for λ and μ, we can write equations (2-1.9) and (2-1.10) in the form

$$\frac{x}{t} = \frac{\mu_i}{\lambda_r\mu_i - \lambda_i\mu_r} \tag{2-2.5}$$

$$\frac{y}{t} = -\frac{\lambda_i}{\lambda_r\mu_i - \lambda_i\mu_r} \tag{2-2.6}$$

We shall now verify that the denominators in these expressions to not vanish in the slit z-plane of figure 8.

We define

$$D(\xi,\eta) = \lambda_r(\xi,\eta)\mu_i(\xi,\eta) - \lambda_i(\xi,\eta)\mu_r(\xi,\eta) \tag{2-2.7}$$

which can be written in the more convenient form

$$D(\xi,\eta) = -(\mu_r^2 + \mu_i^2)\mathrm{Im}\left\{\frac{\lambda(z)}{\mu(z)}\right\} \tag{2-2.8}$$

or, with the aid of (2-1.4) and (2-1.5) we may write

$$D(\xi,\eta) = -(\frac{Aa}{c})|\mu(z)|^2 \text{Im}\left\{\sqrt{\frac{z^2 - c^2}{(z^2 - a^2)(z^2 - A^2)}}\right\} \qquad (2\text{-}2.9)$$

Now $|\mu(z)|^2$ is everywhere positive in the slit z-plane. We will show that the other factor in (2-2.9) never vanishes, by applying the maximum principle for harmonic functions. We excise small regions near $z = \pm a$ and $z = \pm A$, and consider the harmonic function in the resulting domain. Near the points $z = \pm a$ and $z = \pm A$ we have

$$\text{Im}\left\{\sqrt{\frac{z^2 - c^2}{(z^2 - a^2)(z^2 - A^2)}}\right\} < 0 \qquad (2\text{-}2.10)$$

while on the slits we have

$$\text{Im}\left\{\sqrt{\frac{z^2 - c^2}{(z^2 - a^2)(z^2 - A^2)}}\right\} = 0 \qquad (2\text{-}2.11)$$

Since an harmonic function can assume its maximum only on the boundary of a domain, we conclude that (2-2.10) holds throughout the slit z-plane. Coupled with the fact that $|\mu(z)|^2$ is positive, we have

$$D(\xi,\eta) < 0 \qquad (2\text{-}2.12)$$

in the slit z-plane of figure 8. Thus, (2-1.8) defines a differentiable mapping of the z-plane into the $(\frac{x}{t}, \frac{y}{t})$-plane.

We will now show that the mapping is one-one.

To show that the mapping is one-one we must show that for given $\frac{x}{t}$ and $\frac{y}{t}$ there is not more than one value of z such that (2-2.5) and (2-2.6) are satisfied. The possible values of z are the four roots of the quartic equation

$$[\mu(z)(\tfrac{y}{t})]^2 = [1 - \lambda(z)(\tfrac{x}{t})]^2 \tag{2-2.13}$$

This equation always has two real roots, one positive and one negative, such that $c^2 \leq z^2 \leq a^2$. The existence of the negative root is assured by the fact that the family of lines

$$(\tfrac{x}{t}) \pm (\tfrac{c}{aA}) \sqrt{\frac{(z^2 - A^2)(z^2 - a^2)}{z^2 - c^2}}\, (\tfrac{y}{t}) = z \tag{2-2.14}$$

$$(-a \leq z \leq -c)$$

covers the whole $(\frac{x}{t}, \frac{y}{t})$-plane (the $\frac{y}{t}$ intercept of such a line can have all values from $-\infty$ to $+\infty$). A similar observation with $c \leq z \leq a$ guarantees the existence of the positive real root. Thus, (2-2.13) can have only two complex roots which are complex conjugates. By equations (2-2.6) and (2-2.12) we see that

$$\operatorname{sgn}\left\{ \operatorname{Im}(z) \right\} = \operatorname{sgn}\left\{ \frac{y}{t} \right\} \tag{2-2.15}$$

Only one of the two complex-conjugate roots of (2-2.13) will satisfy equation (2-2.15). This shows that the mapping (2-1.18) is one-one, except for the regions $-A < z < -a$, $-c < z < c$, and $a < z < A$. In these cases, $\mu(z)$ is pure imaginary and $\lambda(z)$ is real. From equations (2-1.9) and (2-1.10) we conclude that

$$\xi = z = \frac{x}{t} \tag{2-2.16}$$

which is a unique value of z.

Proof of (ii): This result was proved above, but can be seen directly from equation (2-1.8).

Proof of (iii): A straightforward power series expansion yields

$$\sqrt{\frac{(1 - z^2/A^2)(1 - z^2/a^2)}{1 - z^2/c^2}} = 1 + \frac{z^4}{2a^2A^2} + \frac{z^6}{2a^2A^2c^2} + O(|z|^8) \quad (|z| < c) \tag{2-2.17}$$

Using this result in (2-1.6) then gives the desired relation, namely (2-2.1).

Proof of (iv): With $z = \xi + i\eta$, we consider $\xi < -A$, $-a < \xi < -c$, $c < \xi < a$, or $A < \xi$, and assume that η is small, so that we will be near the boundary of the slit z-plane. With ξ in these intervals and $\eta = 0$, $\lambda(z)$ and $\mu(z)$ are real. Expanding in powers of η we find that the real and imaginary parts of (2-1.8) are

$$\lambda(\xi)(\frac{x}{t}) + \mu(\xi)(\frac{y}{t}) = 1 \qquad (2\text{-}2.18)$$

$$\eta \frac{\partial}{\partial\xi}\left\{\lambda(\xi)(\frac{x}{t}) + \mu(\xi)(\frac{y}{t})\right\} = 0 \qquad (2\text{-}2.19)$$

The sign of μ in these equations will depend on whether $\eta \rightarrow 0$ through positive or negative values. Equations (2-2.18) and (2-2.19) determine the image in the $(\frac{x}{t}, \frac{y}{t})$-plane of the bounding slits in the z-plane. We will write these equations as

$$\lambda(\xi)(\frac{x}{t}) + \mu(\xi)(\frac{y}{t}) = 1 \qquad (2\text{-}2.20)$$

$$\lambda'(\xi)(\frac{x}{t} + \mu'(\xi)(\frac{y}{t}) = 0 \qquad (2\text{-}2.21)$$

where the primes in (2-2.21) denote differentiation with respect to ξ. The determinant of these two equations is not zero. Thus, the point (x,y,t) is on the envelope of the characteristic planes given by

$$\lambda(\xi)x + \mu(\xi)y = t \qquad (2\text{-}2.22)$$

To see that the determinant of equations (2-2.20) and (2-2.21) does not vanish, we write the determinant in the form

$$\lambda\mu' - \lambda'\mu = \lambda^2 \frac{\partial}{\partial\xi}(\frac{\mu}{\lambda}) = \frac{\lambda^3}{2\mu}\frac{\partial}{\partial\xi}[(\frac{\mu}{\lambda})^2] \qquad (2\text{-}2.23)$$

and compute directly that $\frac{\partial}{\partial\xi}[(\mu/\lambda)^2]$ vanishes only for $\xi = 0$ or $\xi = \pm\sqrt{2c}$, and these points are not on the bounding slits.

Proof of (v): The line given by

$$\lambda(\xi)(\frac{\bar{x}}{t}) + \mu(\xi)(\frac{\bar{y}}{t}) = 1 \tag{2-2.24}$$

is tangent to the characteristic conoid at the point

$$(\frac{\bar{x}}{t}) = (\frac{x}{t}) \quad \text{and} \quad (\frac{\bar{y}}{t}) = \frac{y}{t}) \tag{2-2.25}$$

and the intercept of this line on the $(\frac{x}{t})$-axis is at $(\frac{x}{t}) = \frac{1}{\lambda(\xi)} = \xi = z$.

Concisely stated, under the mapping (2-1.8) the right half of the fast wave envelope, BAD (see figure 4), maps onto the slit from A to ∞ (see figure 8). Similarly, the left half of the fast wave envelope, BCD, maps onto the slit from -∞ to -A. The two cusped portions of the slow wave envelope map onto the two slits from -a to -c and c to a. Finally, distances along the undisturbed magnetic axis are preserved, and the mapping approximates the identity mapping in a neighborhood of the origin.

2-3. The Form of the Elliptic Solution

We noted earlier that the fast wave envelope ABCD determines the boundary of the range of influence of an initial disturbance at $(x,y,t) = (0,0,0)$. For this reason, we shall take the solution (ρ,u,v,B_x,B_y) to be zero outside the fast wave envelope. We satisfy the jump relations across the fast wave envelope by taking the solution to be continuous across this characteristic surface. There are three reasons for this choice. First, it is the simplest way to proceed, and if we can construct such a solution, we know by our uniqueness theorem that this is the correct way to proceed. Secondly, Weitzner's fundamental solution [11], which is a homogeneous function of (x,y,t) of degree minus one, strongly suggests this choice. Lastly, Gardner's diffraction problem [5] proceeds in this manner. Thus, in the complex z-plane, we take our solution to be zero on both the upper and lower edges of the slits from $-A$ to $-\infty$ and from A to ∞ (these slits are the image of the fast wave envelope ABCD).

We shall take the solution to be a constant state inside the cusped portions of the slow wave envelope, and again satisfy the jump relations by taking them to be continuous across these portions of the slow wave envelope. (The constant state in cFaE need not be identical with the constant state in GIHK.) These restrictions on the form of the solution are motivated by the same considerations as stated in connection with the nature of the solution across the fast wave envelope. In the complex z-plane, we take the variables to be constants on the slit from -a to -c and (not necessarily the same) constants on the slit from c to a. (These slits are the image of cFaE and GIKH.)

Finally, there may be a contact discontinuity across the straight line segment of the slow wave envelope (Gc) which lies outside the piston. On this line segment the unit normal $\vec{n} = (n_x, n_y, n_t)$ is given by

$$\vec{n} = (0,1,0) \qquad (2\text{-}3.1)$$

Referring to the jump relations (1-4.1) through (1-4.6) we see that we must require that

$$a^2[\rho] + A^2[B_x] = 0 \quad \text{and} \quad [v] = [B_y] = 0 \qquad (2\text{-}3.2)$$

across these two line segments.

We now use the representations (2-1.4) and (2-1.5) in the formal solution given by equations (1-3.14) through (1-3.22), recalling to take the imaginary parts of these integrals. We write $f(\tau)$ for $V(\tau)$ and have

$$\rho(z) = \left(\frac{c}{aA}\right) \mathrm{Im} \int_{\infty}^{z} \sqrt{\frac{\tau^2 - A^2}{(\tau^2 - a^2)(\tau^2 - c^2)}}\ \tau\, f(\tau) d\tau \qquad (2\text{-}3.3)$$

$$u(z) = (\frac{ac}{A}) \operatorname{Im} \int_{\infty}^{z} \sqrt{\frac{\tau^2 - A^2}{(\tau^2 - a^2)(\tau^2 - c^2)}}\, f(\tau)d\tau \qquad (2\text{-}3.4)$$

$$v(z) = \operatorname{Im} \int_{\infty}^{z} f(\tau)d\tau \qquad (2\text{-}3.5)$$

$$B_x(z) = (\frac{c}{aA}) \operatorname{Im} \int_{\infty}^{z} \sqrt{\frac{(\tau^2 - a^2)(\tau^2 - A^2)}{\tau^2 - c^2}}\, \frac{f(\tau)}{\tau} \qquad (2\text{-}3.6)$$

$$B_y(z) = -\operatorname{Im} \int_{\infty}^{z} \frac{f(\tau)}{\tau}\, d\tau \qquad (2\text{-}3.7)$$

The square root in (2-3.6) is defined by equation (2-1.6). The square roots in (2-3.3) and (2-3.4) are defined by the relation

$$\sqrt{\frac{\tau^2 - A^2}{(\tau^2 - a^2)(\tau^2 - c^2)}} = (\frac{1}{\tau^2 - a^2}) \sqrt{\frac{(\tau^2 - a^2)(\tau^2 - A^2)}{\tau^2 - c^2}} \qquad (2\text{-}3.8)$$

We take f(z) to be analytic in the slit z-plane, figure 9, outside the image of the piston. The conditions on the slow and fast wave envelopes (more precisely, on the corresponding slits) are satisfied by taking f(z) to be real for real z outside the piston image. By ∞ in these integrals we mean +∞ on the real axis. To satisfy the conditions on the slits in the left-half plane we must be able to integrate around a large simicircular path from +∞ to -∞ without picking up a contribution to the various integrals. To insure that we can do this we require that

$$f(z) \sim \frac{1}{z^2}\left(1 + O\left(\frac{1}{|z|}\right)\right) \quad \text{for large } |z| \qquad (2\text{-}3.9)$$

An examination of the Laurent series of such a function (about the point ∞) shows that it will be real on the imaginary axis. Finally, we require that any singularities which f(z) may possess lie within, or on the surface of the image of the piston in the z-plane.

3. A Solution Involving only the Complex Characteristics

In this section we shall use the formalism developed in the last section to construct a solution depending only on the complex characteristics. (Other solutions which we will discuss later employ both the complex and the real characteristics.) Such a solution will be possible only for special geometries, and we treat the shape of the piston as one of the variables of the problem. Thus, we are led to consider a free boundary problem.

The free boundary problem is formulated in the second subsection. In the last subsection we carry out the solution of this problem by conformal mapping. It turns out that to first order in the expansion speed of the piston the resulting (unique) piston cross-section is circular. The explicit solution to this problem is exhibited. As mentioned earlier, the linearized problem is not evidently a faithful representation of the nonlinear problem. The presence of a magnetic boundary layer is indicated.

3-1. Formulation of the Problem

In order to find a solution involving only the complex characteristics, we must find an analytic function f(z) which has the properties discussed at the end of the last section, and which, in addition, satisfies the appropriate boundary conditions on the surface of the piston, namely (1-5.11) and (1-5.12). We shall consider slow expansions of the piston, so that the profile in the $(\frac{x}{t}, \frac{y}{t})$-plane (and thus in the z-plane) will be small. From (2-2.1) we see that to fourth order we can write

$$\sqrt{\frac{(1 - z^2/A^2)(1 - z^2/a^2)}{1 - z^2/c^2}} = 1 \quad (|z| \text{ small}) \tag{3-1.1}$$

Using this approximation for small $|z|$ on the image of the piston (in the z-plane), and using (2-3.4) and (2-3.5) we can write

$$u(z) = C_u - a^2 \, \mathrm{Re} \int_{\pm p}^{z} \frac{f(\tau)}{a^2 - \tau^2} \, d\tau \tag{3-1.2}$$

$$v(z) = C_v + \mathrm{Im} \int_{\pm t}^{z} f(\tau) d\tau \tag{3-1.3}$$

where

$$C_u = \left(\frac{ac}{A}\right) \mathrm{Im} \int_{\infty}^{\pm p} \sqrt{\frac{\tau^2 - A^2}{(\tau^2 - a^2)(\tau^2 - c^2)}} \, f(\tau) d\tau \tag{3-1.4}$$

and

$$C_v = \mathrm{Im} \int_{\infty}^{\pm t} f(\tau)d\tau \tag{3-1.5}$$

are constants. The point p is the intercept of the piston with the positive $(\frac{x}{t})$-axis, while t is the intercept with the positive $(\frac{y}{t})$-axis (see figure 9). The various plus and minus signs allow for the different quadrants which may be under consideration.

Since f(z) is to be real for real z (to satisfy the conditions at the slow and fast wave envelopes) and have a second order zero at infinity we have immediately that

$$C_v = 0 \tag{3-1.6}$$

Furthermore, as we mentioned earlier, an examination of the Laurent series about the point infinity for the function f(z) shows that f(z) will also be real for z on the imaginary axis (outside the image of the piston). Thus, we have that

$$C_u = 0 \tag{3-1.7}$$

We now introduce arc length s, along the image of the piston as shown in figure 10. On the image of the piston we write

$$z(s) = \xi(s) + i\eta(s) \tag{3-1.8}$$

If we write

$$\theta(z(s)) = \theta_r(\xi(s),\ \eta(s)) + i\theta_i(\xi(s),\eta(s)) \qquad (3\text{-}1.9)$$

we have that

$$\int_{z_o}^{z(s)} \theta(\tau)d\tau = \int_{s_o}^{s} \Big\{\theta_r(\xi(\sigma),\eta(\sigma))$$

$$+ i\theta_i(\xi(\sigma),\eta(\sigma))\Big\}\Big\{\xi_s(\sigma) + i\eta_s(\sigma)\Big\}\, d\sigma \qquad (3\text{-}1.10)$$

where s_o is the value of arc length corresponding to the point z_o. Using (3-1.10) we conclude that

$$\frac{d}{ds}\left\{\operatorname{Im}\int_{z_o}^{z(s)} \theta(\tau)d\tau\right\} = \theta_r(\xi(s),\eta(s))\eta_s(s)$$

$$+ \theta_i(\xi(s),\eta(s))\xi_s(s) \qquad (3\text{-}1.11)$$

$$\frac{d}{ds}\left\{\operatorname{Re}\int_{z_o}^{z(s)} \theta(\tau)d\tau\right\} = \theta_r(\xi(s),\eta(s))\xi_s(s)$$

$$- \theta_i(\xi(s),\eta(s))\eta_s(s) \qquad (3\text{-}1.12)$$

For a point z(s) on the image of the piston we can now write (3-1.2) and (3-1.3) in the form

$$u(z(s)) = -\ a^2\ \operatorname{Re}\int_{t}^{z(s)} \frac{f(\tau)}{a^2 - \tau^2}\, d\tau \qquad (3\text{-}1.13)$$

$$v(z(s)) = \operatorname{Im} \int_{p}^{z(s)} f(\tau)dt \qquad (3\text{-}1.14)$$

Now, since the z-plane is the image of the $(\frac{x}{t}, \frac{y}{t})$-plane under the mapping (2-1.8) we have that

$$\frac{dU_1}{ds}(P) = \xi_s(P) \qquad (3\text{-}1.15)$$

and

$$\frac{dU_2}{ds}(P) = \eta_s(P) \qquad (3\text{-}1.16)$$

where P is a point on the image of the piston. Differentiating (3-1.13) and (3-1.14), with the aid of (3-1.11) and (3-1.12), we find that (3-1.15) and (3-1.16) are equivalent to the two conditions

$$-a^2F_r(\xi(s),\eta(s))\xi_s(s) + a^2F_i(\xi(s),\eta(s))\eta_s(s) = \xi_s(s) \qquad (3\text{-}1.17)$$

$$f_r(\xi(s),\eta(s))\eta_s(s) + f_i(\xi(s),\eta(s))\xi_s(s) = \eta_s(s) \qquad (3\text{-}1.18)$$

where we have written

$$F(z(s)) = \frac{f(z(s))}{a^2 - z^2(s)} = F_r(\xi(s),\eta(s)) + iF_i(\xi(s),\eta(s)) \qquad (3\text{-}1.19)$$

To order $|z|^2$ we write

$$F(z(s)) = \frac{f(z(s))}{a^2} \tag{3-1.20}$$

so that (3-1.15) and (3-1.16) become

$$-f_r(\xi(s),\eta(s))\xi_s(s) + f_i(\xi(s),\eta(s))\eta_s(s) = \xi_s(s) \tag{3-1.21}$$

$$f_r(\xi(s),\eta(s))\eta_s(s) + f_i(\xi(s),\eta(s))\xi_s(s) = \eta_s(s) \tag{3-1.22}$$

In these relations, we regard not only f(z(s)) but also z(s) as unknown quantities. Thus, to find the allowable piston profiles which will yield flows depending only on the complex characteristics we must solve a free boundary problem. We shall formulate this free boundary problem in the next subsection.

3-2. Formulation of the Free Boundary Problem

We denote by S the slope of the yet-to-be determined image of the piston cross-section. That is

$$S(s) = \frac{\eta_s(s)}{\xi_s(s)} \tag{3-2.1}$$

We can rewrite equations (3-1.21) and (3-1.22) in the form

$$-f_r(\xi(s),\eta(s)) + S(s)f_i(\xi(s),\eta(s)) = 1 \qquad (3\text{-}2.2)$$

$$f_r(\xi(s),\eta(s)) + \left(\frac{1}{S(s)}\right) f_i(\xi(s),\eta(s)) = 1 \qquad (3\text{-}2.3)$$

Equation (3-2.3) gives

$$S(s) = \frac{f_i(\xi(s),\eta(s))}{1 - f_r(\xi(s),\eta(s))} \qquad (3\text{-}2.4)$$

Using the result (3-2.4) in (3-2.2) we find that

$$f_r^2(\xi(s),\eta(s)) + f_i^2(\xi(s),\eta(s)) = 1 \qquad (3\text{-}2.5)$$

We shall use equations (3-2.4) and (3-2.5) to solve our problem.

3-3. The Solution of the Problem

We consider the mapping properties of the function $\zeta = f(z)$. This function maps the region exterior to the piston image in the z-plane onto the unit disc in the ζ-plane, infinity being mapped into the origin (see figure 11). As the image of the piston is traversed in the counterclockwise sense, the unit circle of the ζ-plane is traversed in the clockwise sense. If we write

$$f(z(s)) = e^{i\phi(s)} \tag{3-3.1}$$

then equation (3-2.5) is automatically satisfied. Equation (3-2.4) gives

$$S(s) = \frac{\sin \phi(s)}{1 - \cos \phi(s)} = \cot\left(\frac{\phi(s)}{2}\right) \tag{3-3.2}$$

Thus (see figure 12) we have

$$\tan \psi(s) = \cot\left(\frac{\phi(s)}{2}\right) = \tan\left(\frac{\pi}{2} - \frac{\phi(s)}{2}\right) \tag{3-3.3}$$

so that

$$\psi(s) = \frac{\pi}{2} - \frac{\phi(s)}{2} + n\pi \quad \text{(n an integer)} \tag{3-3.4}$$

From the geometry of figure 12 it is seen that the appropriate choice is n = 0, so that (3-3.4) gives

$$\psi(s) = \frac{\pi}{2} - \frac{\phi(s)}{2} \tag{3-3.5}$$

Referring to equation (3-3.1) we see that

$$\phi(s) = \arg f \tag{3-3.6}$$

Equation (3-3.5) gives

$$\arg f = \pi - 2\psi(s) \tag{3-3.7}$$

We will now compute the change in arg f as we traverse the piston image. It is convenient to start at the bottom of the piston image (see figure 12) where $\psi = 0 = 2\pi$ and proceed counterclockwise. (The corresponding value of $\phi/2$ is $3\pi/2$, and the unit circle is traversed in the clockwise sense.) We find that

$$\Delta \arg f = -4\pi \tag{3-3.8}$$

We have traversed the piston image in the negative sense (we are considering the domain exterior to the piston image) and so we have, by the argument principle, that

$$N - P = 2 \tag{3-3.9}$$

where N is the number of zeros of f in the domain exterior to the piston image, P being the corresponding number of poles. Since the function f is to be regular in this exterior domain, $P = 0$, and then (3-3.9) gives $N = 2$. Equation (2-3.9) requires that f have a double zero at infinity. Thus, we can write

$$f(z) = \frac{F(z)}{z^2} \tag{3-3.10}$$

where F(z) is regular in the exterior domain of the z-plane and free of zeros. Now, we can introduce the square root of the analytic function f(z), without needing a branch cut.

We write

$$g(z) = \sqrt{f(z)} = \frac{\sqrt{F(z)}}{z} \tag{3-3.11}$$

Equations (3-2.4) and (3-2.5), with the aid of (3-3.1) and (3-3.2) give

$$S(s) = \frac{g_r(\xi(s),\eta(s))}{g_i(\xi(s),\eta(s))} \tag{3-3.12}$$

$$g_r^2(\xi(s),\eta(s)) + g_i^2(\xi(s),\eta(s)) = 1 \tag{3-3.13}$$

The function g(z) has a first order zero at infinity in the z-plane. The point infinity in the plane, under the mapping w = g(z), maps into the origin in the w-plane.

We denote the inverse mapping by

$$z = k(w) \tag{3-3.14}$$

The function k(w) will have a first order pole at the origin (see figure 13). We now exploit one of the principles of conformal mapping: if the mapping z = Z(w) maps a curve Γ_w in the w-plane onto a curve Γ_z in the z-plane, and the tangent to Γ_w makes an angle γ_w with the real axis in the w-plane, while the tangent to Γ_z makes an angle γ_z with the real axis in the z-plane, then

$$\arg\left[\frac{dZ(w)}{dz}\right] = \gamma_z - \gamma_w \tag{3-3.15}$$

Using (3-3.5) we have

$$\arg k'(w) = \left(\frac{\pi}{2} - \frac{\phi(s)}{2}\right) - \left(\frac{\phi(s)}{2} - \frac{\pi}{2}\right) = \pi - \phi(s) \tag{3-3.16}$$

Now,

$$\arg w = \frac{\phi(s)}{2} \tag{3-3.17}$$

so that

$$\arg w^2 = \phi(s) \tag{3-3.18}$$

Adding (3-3.16) and (3-3.18) we find that

$$\arg w^2 + \arg k'(w) = \pi \tag{3-3.19}$$

Therefore

$$\arg[w^2 k'(w)] = (2m + 1)\pi \quad (m \text{ an integer}) \tag{3-3.20}$$

We draw the conclusion that $w^2k'(w)$ is negative real on the unit circle in the w-plane.

As we noted above, $k(w)$ will have a first order pole at the origin, and be regular elsewhere inside and on the unit circle. The derivative $k'(w)$ will have the same regularity properties, but will have a second order pole at the origin. Thus, $w^2k'(w)$ will be regular on the unit disc in the w-plane, and real on the unit circle. We can continue $w^2k'(w)$ by the Schwarz reflection principle to a function that is regular in the entire w-plane. By Liouville's theorem, $w^2k'(w)$ must be a constant, in fact, a negative real constant by virtue of (3-3.20). Thus we may write

$$w^2k'(w) = -r_o \quad (r_o \text{ a positive constant}) \tag{3-3.21}$$

If we solve this differential equation for the function $k(w)$ we find that

$$k(w) = \frac{r_o}{w} \tag{3-3.22}$$

By equation (3-3.14) we finally have

$$z = \frac{r_o}{w} \tag{3-3.23}$$

Multiplying equation (3-3.23) by its complex conjugate we find that

$$|z|^2 = \frac{r_o^2}{|w|^2} \tag{3-3.24}$$

The inverse mapping $z = k(w)$ maps the unit circle in the w-plane onto the image of the piston in the z-plane. Setting $|w|^2 = 1$ in equation (3-3.24) we find that

$$|z|^2 = r_o^2 \tag{3-3.25}$$

which is the equation of the image of the piston in the z-plane. From (3-3.23), and the fact that the z-plane is mapped into the w-plane by the function w = g(z), we find that

$$g(z) = \frac{r_o}{z} \tag{3-3.26}$$

Finally, with the aid of equation (3-3.11) we find that

$$f(z) = \frac{r_o^2}{z^2} \tag{3-3.27}$$

which obviously satisfies all the desired conditions.

We thus have that for a sufficiently small circular piston of radius r_o the solution is given by

$$\rho(z) = \left(\frac{cr_o^2}{aA}\right) \operatorname{Im} \int_{\infty}^{z} \sqrt{\frac{\tau^2 - A^2}{(\tau^2 - a^2)(\tau^2 - c^2)}}\ \frac{d\tau}{\tau} \tag{3-3.28}$$

$$u(z) = \left(\frac{acr_o^2}{A}\right) \operatorname{Im} \int_{\infty}^{z} \sqrt{\frac{\tau^2 - A^2}{(\tau^2 - a^2)(\tau^2 - c^2)}}\ \frac{d\tau}{\tau^2} \tag{3-3.29}$$

$$v(z) = (r_o^2) \operatorname{Im} \int_{\infty}^{z} \frac{d\tau}{\tau^2} \tag{3-3.30}$$

$$B_x(z) = \left(\frac{cr_o^2}{aA}\right) \operatorname{Im} \int_{\infty}^{z} \sqrt{\frac{(\tau^2 - a^2)(\tau^2 - A^2)}{\tau^2 - c^2}}\ \frac{d\tau}{\tau^3} \tag{3-3.31}$$

$$B_y(z) = (-r_o^2)\ \mathrm{Im} \int_{\infty}^{z} \frac{d\tau}{\tau^3} \tag{3-3.32}$$

where

$$z = z\left(\frac{x}{t}, \frac{y}{t}\right) \tag{3-3.33}$$

It is interesting to note that the solution is "scaled" according to the size of the piston, but is otherwise independent of the shape. Near the piston, the magnetic field perturbations are of O(1). This indicates a magnetic boundary layer. The field near the surface of the piston is strongly distorted so that it will be tangent to the surface of the piston. This feature of the solution is not known a priori from the linearized equations or boundary conditions.

4. The Hyperbolic Wave

In this section we discuss the hyperbolic wave. This part of the solution is constructed using the two real characteristics from the piston to the characteristic conoid. The first subsection discusses the form of the solution. This subsection is a straightforward extension of the corresponding section which discussed the elliptic solution.

The second subsection examines the nature of this solution. In particular, we point out why a solution involving only the real characteristics is not possible. The final subsection discusses the dependence of ζ, the $\frac{x}{t}$-intercept of a real characteristic, upon the variables $\frac{x}{t}$ and $\frac{y}{t}$. This result will play an important role in some later investigations.

4-1. The Form of the Solution

In analogy with (2-1.4) and (2-1.5) we write for the reciprocal $\frac{x}{t}$ and $\frac{y}{t}$ intercepts of a real characteristic

$$\lambda(\zeta) = \frac{1}{\zeta} \tag{4-1.1}$$

$$\mu(\zeta) = \left(\frac{c}{aA}\right) \frac{1}{\zeta} \sqrt{\frac{(\zeta^2 - A^2)(\zeta^2 - a^2)}{\zeta^2 - c^2}} \tag{4-1.2}$$

Here ζ is a real variable. Geometrically, ζ is the $(\frac{x}{t})$-intercept of any one of the real characteristics from the piston to the characteristic conoid (see figure 14). The two possible values of ζ are the two real roots of the quartic

$$[\mu(\zeta)(\frac{y}{t})]^2 = [1 - \lambda(\zeta)(\frac{x}{t})]^2 \qquad (4\text{-}1.3)$$

We shall assume that the piston is symmetric about both the $\frac{x}{t}$ and $\frac{y}{t}$ axes. The symmetry about the $\frac{x}{t}$-axis (the magnetic axis) is inherent in the problem, and an essential requirement for solutions to exist. The symmetry about the $(\frac{y}{t})$-axis is assumed for the sake of convenience.If this symmetry were not present, the lower limits in the integrals which express the solution would be different in magnitude for the right and left hand hyperbolic waves.

We now introduce the notion of a relative parity diagram. Such a figure consists of a small picture of the axes, with the relative signs of the function in question placed in the four quadrants. We arbitrarily assign the positive sign to the first quadrant. Recalling that $\frac{1}{\mu(\zeta)}$ is the $(\frac{y}{t})$-intercept of the real characteristic whose $(\frac{x}{t})$-intercept is ζ, we can immediately write down the relative parity diagram for the function $\mu(\zeta)$; it is

$$\begin{array}{c|c} + & + \\ \hline - & - \end{array}$$

If we now notice that the factor $\frac{1}{\zeta}$ in (4-1.2) has the relative parity

$$\begin{array}{c|c} - & + \\ \hline - & + \end{array}$$

we can conclude that the square root in (4-1.2) must be defined to have the relative parity

$$\begin{array}{c|c} - & + \\ \hline + & - \end{array}$$

This definition of the square root serves to make $\mu(\zeta)$ positive for the real characteristics in the upper half plane, and negative for the characteristics in the lower half plane.

We now define the $(\frac{x}{t})$-intercept of the real characteristic from the top (or bottom) of the piston to the right hand cusped portion of the slow wave envelope to be ζ_o (see figure 14), and choose $z_o = \pm \zeta_o$ in the formal representation of a conical solution given by equations (1-3.14) through (1-3.18). The plus sign corresponds to the right hand hyperbolic wave while the minus sign corresponds to the left hand hyperbolic wave. As mentioned earlier, if the piston were not symmetric about the $(\frac{y}{t})$-axis, we would have to introduce ζ_o^R and ζ_o^L for the two sides. We can now write the hyperbolic wave in the form

$$\rho(\zeta) = \left\{ \left(\frac{c}{aA}\right) \int_{\pm\zeta_o}^{\zeta} \sqrt{\frac{\tau^2 - A^2}{(\tau^2 - a^2)(\tau^2 - c^2)}}\, \tau g(\tau) d\tau \right\} K\left(\frac{x}{t}, \frac{y}{t}\right) \qquad (4\text{-}1.4)$$

$$u(\zeta) = \left\{ \left(\frac{ac}{A}\right) \int_{\pm\zeta_o}^{\zeta} \sqrt{\frac{\tau^2 - A^2}{(\tau^2 - a^2)(\tau^2 - c^2)}}\, g(\tau) d\tau \right\} K\left(\frac{x}{t}, \frac{y}{t}\right) \qquad (4\text{-}1.5)$$

$$v(\zeta) = \left\{ \int_{\pm\zeta_o}^{\zeta} g(\tau) d\tau \right\} K\left(\frac{x}{t}, \frac{y}{t}\right) \qquad (4\text{-}1.6)$$

$$B_x(\zeta) = \left\{ \left(\frac{c}{aA}\right) \int_{\pm\zeta_o}^{\zeta} \sqrt{\frac{(\tau^2 - a^2)(\tau^2 - A^2)}{\tau^2 - c^2}}\, g(\tau) d\tau \right\} K\left(\frac{x}{t}, \frac{y}{t}\right) \qquad (4\text{-}1.7)$$

$$B_y(\zeta) = \left\{ - \int_{\pm\zeta_o}^{\zeta} \frac{g(\tau)}{\tau}\, d\tau \right\} K\left(\frac{x}{t}, \frac{y}{t}\right) \qquad (4\text{-}1.8)$$

where

$$\zeta = \zeta\left(\frac{x}{t}, \frac{y}{t}\right) \qquad (4\text{-}1.9)$$

The plus signs in the lower limits of integration refer to the right half plane, while the minus signs refer to the left half plane. The function $K(\frac{x}{t}, \frac{y}{t})$ is defined to be +1 in the shaded regions of figure 15 and zero elsewhere. Figure 16 is an enlargement of this shaded region around the cusp of the slow wave envelope which is on the positive $(\frac{x}{t})$-axis. The function $g(\zeta)$ is a real function. We take the relative parity of this function to be

$$\begin{array}{c|c} - & + \\ \hline + & - \end{array}$$

Heuristically, we can argue that this is the correct choice. It is reasonable to suspect that v will be positive in the upper half plane and negative in the lower half plane. An examination of (4-1.6) shows that our choice for the relative parity of $g(\zeta)$ achieves this, provided that the actual sign of $g(\zeta)$ in the first quadrant is negative. In the next section this choice of parity will be rigorously justified.

4-2. The Nature of the Solution

Each characteristic from the piston to the cusped portion of the slow wave envelope determines a value of ζ. Thus, the solution we have written down is constant along each of the real characteristics. In several respects it is analogous to a simple wave solution. By choosing the lower limits of integration in (4-1.4) through (4-1.8) to be $\pm \zeta_o$ we have insured that the variables are zero on the straight line portions of the boundary of the hyperbolic wave; consequently, no problems of discontinuity arise on these portions of the boundary of the hyperbolic wave (see figure 16).

As mentioned above, each of the variables is constant on a given characteristic. Actually, such a characteristic extends to infinity, but we do not want any of the variables to be nonzero outside of the fast wave envelope, since this envelope determines the range of influence. Thus, at some point on each characteristic the values of the variables must abruptly change from constants to zero, i.e., there will be discontinuities in the hyperbolic wave. These discontinuities must occur on a characteristic surface. The only point inside the fast wave envelope where the real characteristics are tangent to a characteristic surface is where they meet the slow wave envelope. For this reason we have introduced the factor $K(\frac{x}{t}, \frac{y}{t})$ which makes all the variables zero beyond the point of tangency on the cusped portion of the slow wave envelope.

Across that portion of the slow wave envelope which is shown in figure 16 (but not dotted) the hyperbolic wave is discontinuous, and does not satisfy the jump relations. It is for this reason that no solution involving only the real characteristics is possible.

4-3. The Dependence of ζ on $\frac{x}{t}$ and $\frac{y}{t}$

We know from the first section that λ and μ satisfy the following two equations:

$$t - \lambda(\zeta)x - \mu(\zeta)y = 0 \tag{4-3.1}$$

$$1 - (a^2 + A^2)(\lambda^2(\zeta) + \mu^2(\zeta)) + a^2A^2\lambda^2(\zeta)(\lambda^2(\zeta) + \mu^2(\zeta)) = 0 \tag{4-3.2}$$

Equation (4-3.1) gives $\lambda(\zeta)$ as a function of $\mu(\zeta)$ and $\frac{x}{t}$ and $\frac{y}{t}$, namely

$$\lambda(\zeta) = \frac{1}{\zeta} = \frac{1 - \mu(\zeta)Y}{X} \tag{4-3.3}$$

where we have written

$$X = \frac{x}{t} \tag{4-3.4}$$

$$Y = \frac{y}{t} \tag{4-3.5}$$

Inserting the representation (4-3.3) into equation (4-3.2), we obtain, after some manipulations, an equation relating ζ and $\frac{x}{t}$ and $\frac{y}{t}$:

$$(Y^2 - (a^2 + A^2))\zeta^4 + 2(a^2 + A^2)X\zeta^3 + (a^2A^2 - (a^2 + A^2)(X^2 + Y^2))\zeta^2 - 2a^2A^2X\zeta + a^2A^2(X^2 + Y^2) = 0 \tag{4-3.6}$$

where

$$\zeta = \zeta(X,Y) \tag{4-3.7}$$

We now obtain ζ as a power series in X and Y by employing the two-dimensional form of Taylor's theorem, namely

$$\zeta(X,Y) = \zeta(0.0) + \zeta_X(0,0)X + \zeta_Y(0,0)Y \tag{4-3.8}$$
$$+ \frac{1}{2}\zeta_{XX}(0,0)X^2 + \zeta_{XY}(0.0)XY + \frac{1}{2}\zeta_{YY}(0,0)Y^2 + \ldots$$

By implicit differentiation of equation (4-3.6) we find that

$$\frac{\partial\zeta}{\partial X}(0,0) = \pm c \tag{4-3.9}$$

$$\frac{\partial\zeta}{\partial Y}(0,0) = 0 \tag{4-3.10}$$

$$\frac{\partial^2\zeta}{\partial X^2}(0,0) = 0 \tag{4-3.11}$$

$$\frac{\partial^2\zeta}{\partial X\partial Y}(0.0) = 0 \tag{4-3.12}$$

$$\frac{\partial^2\zeta}{\partial Y^2}(0,0) = \pm \frac{c^3}{a^2A^2} \tag{4-3.13}$$

Thus, using equations (4-3.9) through (4-3.13) in equation (4-3.8) we obtain

$$\zeta(X,Y) = \pm\, c[1 + \frac{c^2 Y^2}{2a^2 A^2} + \ldots] \qquad (4\text{-}3.14)$$

or, using (4-3.4) and (4-3.5), we have

$$\zeta(\frac{x}{t}, \frac{y}{t}) = \pm\, c[1 + \frac{c^2}{2a^2 A^2}(\frac{y}{t})^2 + \ldots] \qquad (4\text{-}3.15)$$

5. The Solution for a Symmetric Piston with a Full Hyperbolic Wave

In this section we shall construct the solution for an arbitrary piston whose cross-section is symmetric about both the $(\frac{x}{t})$ and $(\frac{y}{t})$-axes, although we briefly indicate how the solution might be obtained for a piston symmetric about the $(\frac{x}{t})$-axis only, when such a solution is possible. We further suppose that the entire surface is covered by the hyperbolic wave; this simply says that the top and bottom of the piston are not too flat. This latter restriction will be dropped in the next section. The solution is constructed by employing both the real and complex characteristics.

In the first subsection we write down the formal solution which we shall use. The second subsection discusses the boundary conditions which we shall impose on this formal solution. Roughly speaking, there are two conditions which are not immediately satisfied; these are the conditions on part of the slow wave envelope, and the conditions on the piston. The third and fourth subsections discuss how we satisfy these remaining conditions.

The next three subsections discuss the solution of the relevant boundary value problem. Various singularities may appear in the solution, and the nature of these singularities is examined. Subsection eight discusses the determination of the one arbitrary constant which occurs in the solution.

The section ends with a discussion of the possible extension of our methods to a piston whose cross-section is not symmetric about the $(\frac{y}{t})$-axis.

5-1. The Form of the Solution

We write our solution as the sum of an elliptic solution and a hyperbolic wave. Using equations (2-3.3) through (2-3.7) and equations (4-1.4) through (4-1.8) we can write our formal solution in the form

$$\rho(\frac{x}{t}, \frac{y}{t}) = (\frac{c}{aA}) \ \mathrm{Im} \int_{\infty}^{z} \sqrt{\frac{\tau^2 - A^2}{(\tau^2 - a^2)(\tau^2 - c^2)}} \ \tau f(\tau) d\tau \qquad (5\text{-}1.1)$$

$$+ \left\{ (\frac{c}{aA}) \int_{\pm\zeta_0}^{\zeta} \sqrt{\frac{\tau^2 - A^2}{(\tau^2 - a^2)(\tau^2 - c^2)}} \ \tau g(\tau) d\tau \right\} K(\frac{x}{t}, \frac{y}{t})$$

$$u(\frac{x}{t}, \frac{y}{t}) = (\frac{ac}{A}) \ \mathrm{Im} \int_{\infty}^{z} \sqrt{\frac{\tau^2 - A^2}{(\tau^2 - a^2)(\tau^2 - c^2)}} \ f(\tau) d\tau \qquad (5\text{-}1.2)$$

$$+ \left\{ (\frac{ac}{A}) \int_{\pm\zeta_0}^{\zeta} \sqrt{\frac{\tau^2 - A^2}{(\tau^2 - a^2)(\tau^2 - c^2)}} \ g(\tau) d\tau \right\} K(\frac{x}{t}, \frac{y}{t})$$

$$v(\tfrac{x}{t}, \tfrac{y}{t}) = \mathrm{Im} \int_{\infty}^{z} f(\tau)d\tau + \left\{ \int_{\pm\zeta_o}^{\zeta} g(\tau)d\tau \right\} K(\tfrac{x}{t}, \tfrac{y}{t}) \qquad (5\text{-}1.3)$$

$$B_x(\tfrac{x}{t}, \tfrac{y}{t}) = (\tfrac{c}{aA})\ \mathrm{Im} \int_{\infty}^{z} \sqrt{\frac{(\tau^2 - a^2)(\tau^2 - A^2)}{\tau^2 - c^2}}\ \frac{f(\tau)}{\tau}\, d\tau \qquad (5\text{-}1.4)$$

$$+ \left\{ (\tfrac{c}{aA}) \int_{\pm\zeta_o}^{\zeta} \sqrt{\frac{(\tau^2 - a^2)(\tau^2 - A^2)}{\tau^2 - c^2}}\ \frac{g(\tau)}{\tau}\, d\tau \right\} K(\tfrac{x}{t}, \tfrac{y}{t})$$

$$B_y(\tfrac{x}{t}, \tfrac{y}{t}) = -\mathrm{Im} \int_{\infty}^{z} \frac{f(\tau)}{\tau}\, d\tau - \left\{ \int_{\pm\zeta_o}^{\zeta} \frac{g(\tau)}{\tau}\, d\tau \right\} K(\tfrac{x}{t}, \tfrac{y}{t}) \qquad (5\text{-}1.5)$$

where

$$z = z(\tfrac{x}{t}, \tfrac{y}{t}) \qquad (5\text{-}1.6)$$

and

$$\zeta = \zeta(\tfrac{x}{t}, \tfrac{y}{t}) \qquad (5\text{-}1.7)$$

As before, $f(z)$ is an analytic function of the complex variable z, and $g(\zeta)$ is a real function of the real variable ζ. The function $K(\frac{x}{t}, \frac{y}{t})$ is again defined to be +1 in the shaded region of figure 15, and zero elsewhere. In the real integrals, the plus signs in the lower limits of integration refer to the right half plane; the munus signs refer to the left half plane.

To satisfy the boundary and matching conditions, which we shall discuss in the next subsection, we will place certain restrictions on f and the path of integration in the contour integrals. The boundary conditions on the surface of the piston serve to complete the determination of $f(z)$ and $g(\zeta)$.

5-2. The Boundary Conditions

As before, we take the solution to be zero outside the fast wave envelope ABCD, thus preserving the range of influence. Proceeding as before, we take the solution to be continuous across that characteristic surface. Because of the factor $K(\frac{x}{t}, \frac{y}{t})$ which multiplies the real integrals, the solution in the neighborhood of the fast wave envelope, ABCD, consists only of the contour integrals. We satisfy these requirements by taking the analytic function $f(z)$ to be real on the slits from $-A$ to $-\infty$ and from A to ∞ (see figure 9). The function $f(z)$ must also have a double zero at infinity.

Inside that portion of the cusped slow wave envelopes which is not covered by the hyperbolic wave (see figure 16) the solution still consists of only the elliptic solution. Again, we require the solution to be a constant state in these regions, and take it to be continuous across the bounding portion of the slow wave envelope. This condition is satisfied by taking $f(z)$ to be real on the real axis from $-A$ to $-\zeta_o$ and from ζ_o to A. Coupled with the fast wave envelope conditions, we now have that $f(z)$ must be real for z real and $-\infty < z \leq -\zeta_o$ or $\zeta_o \leq z < \infty$. Furthermore

$$f(z) = \frac{1}{z^2}\left(1 + 0\left(\frac{1}{|z|}\right)\right) \qquad (|z| \text{ large}) \tag{5-2.1}$$

That portion of the cusped slow wave envelopes across which the hyperbolic wave exhibits a discontinuity maps (under (2-1.8)) onto the two slits from $-\zeta_o$ to $-c$ and from ζ_o to c. (These are really pieces of the slits from $-a$ to $-c$ and c to a; see figure 17.) We shall take the solution to be continuous across this portion of the slow envelope also, but here both the elliptic and hyperbolic portions of the solution must be considered. Thus, we require that the elliptic solution on the slits from $-\zeta_o$ to $-c$ and from c to ζ_o match the sum of the hyperbolic wave and the constant state in the other portions of the slow wave envelope. We shall now state this mathematically.

The constant state is given by

$$\rho_c^{\pm}\left(\frac{x}{t}, \frac{y}{t}\right) = \left(\frac{c}{aA}\right) \operatorname{Im} \int_{\pm\infty}^{\pm\zeta_o} \sqrt{\frac{\tau^2 - A^2}{(\tau^2 - a^2)(\tau^2 - c^2)}}\; \tau f(\tau)\, d\tau \tag{5-2.2}$$

$$u_c^{\pm}\left(\frac{x}{t}, \frac{y}{t}\right) = \left(\frac{ac}{A}\right) \operatorname{Im} \int_{\pm\infty}^{\pm\zeta_o} \sqrt{\frac{\tau^2-A^2}{(\tau^2-a^2)(\tau^2-c^2)}}\, f(\tau)d\tau \qquad (5\text{-}2.3)$$

$$v_c^{\pm}\left(\frac{x}{t}, \frac{y}{t}\right) = \operatorname{Im} \int_{\pm\infty}^{\pm\zeta_o} \sqrt{\frac{\tau^2-A^2}{(\tau^2-a^2)(\tau^2-c^2)}}\, f(\tau)d\tau \qquad (5\text{-}2.4)$$

$$B_{xc}^{\pm}\left(\frac{x}{t}, \frac{y}{t}\right) = \left(\frac{c}{aA}\right) \operatorname{Im} \int_{\pm\infty}^{\pm\zeta_o} \sqrt{\frac{(\tau^2-a^2)(\tau^2-A^2)}{\tau^2-c^2}}\, \frac{f(\tau)}{\tau}\, d\tau \qquad (5\text{-}2.5)$$

$$B_{yc}^{\pm}\left(\frac{x}{t}, \frac{y}{t}\right) = -\operatorname{Im} \int_{\pm\infty}^{\pm\zeta_o} \frac{f(\tau)}{\tau}\, d\tau \qquad (5\text{-}2.6)$$

In these formulas the plus signs refer to the right half plane; the minus signs refer to the left half plane. It should be noted that the only contribution to these integrals occurs from ±A to ±a. The total solution will be continuous across the slow wave envelope if we satisfy the following five conditions:

$$\rho_c^{\pm}\left(\frac{x}{t}, \frac{y}{t}\right) + \left(\frac{c}{aA}\right) \int_{\pm\zeta_o}^{\zeta} \sqrt{\frac{\tau^2-A^2}{(\tau^2-a^2)(\tau^2-c^2)}}\, \tau g(\tau)d\tau$$

$$= \left(\frac{c}{aA}\right) \operatorname{Im} \int_{\pm\infty}^{\zeta} \sqrt{\frac{\tau^2-A^2}{(\tau^2-a^2)(\tau^2-c^2)}}\, \tau f(\tau)d\tau \qquad (5\text{-}2.7)$$

$$u_c^{\pm}(\tfrac{x}{t},\ \tfrac{y}{t}) + (\tfrac{ac}{A}) \int_{\pm\zeta_o}^{\zeta} \sqrt{\frac{\tau^2-A^2}{(\tau^2-a^2)(\tau^2-c^2)}}\ g(\tau)d\tau$$

$$= (\tfrac{ac}{A})\ \mathrm{Im} \int_{\pm\infty}^{\zeta} \sqrt{\frac{\tau^2-A^2}{(\tau^2-a^2)(\tau^2-c^2)}}\ f(\tau)d\tau \qquad (5\text{-}2.8)$$

$$v_c^{\pm}(\tfrac{x}{t},\ \tfrac{y}{t}) + \int_{\pm\zeta_o}^{\zeta} g(\tau)d\tau = \mathrm{Im} \int_{\pm\infty}^{\zeta} f(\tau)d\tau \qquad (5\text{-}2.9)$$

$$B_{xc}^{\pm}(\tfrac{x}{t},\ \tfrac{y}{t}) + (\tfrac{c}{aA}) \int_{\pm\zeta_o}^{\zeta} \sqrt{\frac{(\tau^2-a^2)(t^2-A^2)}{\tau^2-c^2}}\ \frac{g(\tau)}{\tau}\ d\tau$$

$$= (\tfrac{c}{aA})\ \mathrm{Im} \int_{\pm\infty}^{\zeta} \sqrt{\frac{(\tau^2-a^2)(\tau^2-A^2)}{\tau^2-c^2}}\ \frac{f(\tau)}{\tau}\ d\tau \qquad (5\text{-}2.10)$$

$$B_{yc}^{\pm}(\tfrac{x}{t},\ \tfrac{y}{t}) - \int_{\pm\zeta_o}^{\zeta} \frac{g(\tau)}{\tau}\ d\tau = -\ \mathrm{Im} \int_{\pm\infty}^{\zeta} \frac{f(\tau)}{\tau}\ d\tau \qquad (5\text{-}2.11)$$

where

$$\zeta = \zeta(\tfrac{x}{t},\ \tfrac{y}{t}) \qquad (5\text{-}2.12)$$

and

$$c \leq |\zeta| \leq \zeta_o \qquad (5\text{-}2.13)$$

If we now use equations (5-2.2) through (5-2.6) we see that the conditions (5-2.7) through (5-2.11) can be written in the form

$$\left(\frac{c}{aA}\right)\int_{\pm\zeta_0}^{\zeta}\sqrt{\frac{\tau^2-A^2}{(\tau^2-a^2)(\tau^2-c^2)}}\;\tau(g(\tau)-\mathrm{Im}\,f(\tau))d\tau=0 \qquad (5\text{-}2.14)$$

$$\left(\frac{ac}{A}\right)\int_{\pm\zeta_0}^{\zeta}\sqrt{\frac{\tau^2-A^2}{(\tau^2-a^2)(\tau^2-c^2)}}\;(g(\tau)-\mathrm{Im}\,f(\tau))d\tau=0 \qquad (5\text{-}2.15)$$

$$\int_{\pm\zeta_0}^{\zeta}(g(\tau)-\mathrm{Im}\,f(\tau))d\tau=0 \qquad (5\text{-}2.16)$$

$$\left(\frac{c}{aA}\right)\int_{\pm\zeta_0}^{\zeta}\sqrt{\frac{(\tau^2-a^2)(\tau^2-A^2)}{\tau^2-c^2}}\;(g(\tau)-\mathrm{Im}\,f(\tau))d\tau=0 \qquad (5\text{-}2.17)$$

$$-\!\!\!\!\!\!\int_{\pm\zeta_0}^{\zeta}(g(\tau)-\mathrm{Im}\,f(\tau))\,\frac{d\tau}{\tau}=0 \qquad (5\text{-}2.18)$$

We can satisfy equations (5-2.14) through (5-2.18) by taking

$$g(\xi)=\mathrm{Im}\,f(\xi) \qquad c\le|\xi|\le\zeta_0 \qquad (5\text{-}2.19)$$

We shall discuss this equation in some detail in the next subsection.

On that portion of the line segment Gc which lies outside the piston we again allow for a contact discontinuity, by requiring that

$$a^2[\rho] + A^2[B_x] = 0 \quad [v] = [B_y] = 0 \tag{5-2.20}$$

across those line segments. In connection with this possible contact discontinuity, we restrict the path of integration in the contour integrals to cross the real axis only in the two intervals (-A,-a) and (a,A).

Finally, we require that at a point P on the surface of the piston

$$u(P) = U_1(P) \tag{5-2.21}$$

$$v(P) = U_2(P) \tag{5-2.22}$$

To sum up, we require that f(z) be real for z real and $\zeta_o \leq |z| \leq \infty$, and have a double zero at infinity. The path of integration in the contour integrals is restricted as mentioned above. Finally, we require that (5-2.19) through (5-2.22) be satisfied.

5-3. The Matching on the Slow Wave Envelope

In this subsection we will discuss how we satisfy equation (5-2.19). We will denote the order of the expansion speed of the piston by ε; more precisely, the maximum of the y-component of the expansion velocity of the piston is denoted by ε. Then, by (4-3.15) we have

$$\zeta_o = c + O(\varepsilon^2) \tag{5-3.1}$$

Thus, the length of the interval (c,ζ_o) is of order ε^2. The solution to our problem is nonuniform in ε. In other words, the problem is inherently of lower order in the neighborhood of the tips of the cusped portions of the slow wave envelope. Elsewhere, the first order problem is of the same character as the second order problem, but in these small regions the first order problem is of an entirely different nature than the second order problem. Thus, in these small regions, we shall carry the solution to second order in the piston velocity. Recalling the relative parity of the function $g(\zeta)$, we can see from equation (5-2.19) that it will be helpful to write

$$f(z) = \left\{ \frac{2}{\pi} \int_c^{\zeta_o} \frac{\xi r(\xi)}{\xi^2 - z^2}\, d\xi \right\} F(z) \tag{5-3.2}$$

The expression inside the bracket can be written as the sum of two singular integrals. If we take $r(\xi)$ to be an even real function, we have

$$\frac{2}{\pi}\int_{c}^{\zeta_o} \frac{\xi r(\xi)}{\xi^2 - z^2}\, d\xi = \frac{1}{\pi}\int_{c}^{\zeta_o} \frac{r(\xi)}{\xi - z}\, d\xi - \frac{1}{\pi}\int_{-\zeta_o}^{-c} \frac{r(\xi)}{\xi - z}\, d\xi \qquad (5\text{-}3.3)$$

Thus, the factor in the brackets will take care of the discontinuities of $f(z)$ across the slits from $-\zeta_o$ to c and from c to ζ_o. Furthermore, for large $|z|$, this factor behaves like $1/z^2$, so we need only require that the analytic function $F(z)$ be bounded at infinity. Thus, in (5-3.2), $F(z)$ is an analytic function, outside the piston image, which is bounded at infinity and real for real z. The function r will be chosen to satisfy equation (5-2.19). For future convenience, we normalize the function r by setting

$$\int_{c}^{\zeta_o} \frac{r(\xi)}{\xi} = \frac{\pi}{2} \qquad (5\text{-}3.4)$$

We now examine these singular integrals, making use of the Plemelj formulas. We denote the upper half plane limit of a function by a superscript plus, appending the superscript minus to denote the lower half plane limit. Let

$$R_1(z) = \frac{1}{\pi}\int_{c}^{\zeta_o} \frac{r(\xi)}{\xi - z}\, d\xi \qquad (5\text{-}3.5)$$

For $z = \zeta_1$, with $c < \zeta_1 < \zeta_o$, the Plemelj formulas give

$$R_1^+(\zeta_1) = ir(\zeta_1) + \frac{P}{\pi}\int_c^{\zeta_0} \frac{r(\xi)}{\xi-\zeta_1}\, d\xi \tag{5-3.6}$$

and

$$R_1^-(\zeta_1) = -ir(\zeta_1) + \frac{P}{\pi}\int_c^{\zeta_0} \frac{r(\xi)}{\xi-\zeta_1}\, d\xi \tag{5-3.7}$$

where the P preceding an integral indicates that the integral is to be interpreted in the Cauchy principal value sense.

Similarly, if we define

$$R_2(z) = \frac{1}{\pi}\int_{-\zeta_0}^{-c} \frac{r(\xi)}{\xi - z}\, d\xi \tag{5-3.8}$$

we have for $z = \zeta_2$, $-\zeta_0 < \zeta_2 < -c$, that

$$R_2^+(\zeta_2) = ir(\zeta_2) + \frac{P}{\pi}\int_{-\zeta_0}^{-c} \frac{r(\xi)}{\xi-\zeta_2}\, d\xi \tag{5-3.9}$$

and

$$R_2^-(\zeta_2) = -ir(\zeta_2) + \frac{P}{\pi}\int_{-\zeta_0}^{-c} \frac{r(\xi)}{\xi-\zeta_2}\, d\xi \tag{5-3.10}$$

Thus, bearing in mind the relative parity of the function $g(\zeta)$, and noticing that the principal value integrals are real, we can satisfy equation (5-2.19) by requiring that

$$r(\zeta)F(\zeta) = g(\zeta^+) \qquad c < |\zeta| < \zeta_0 \tag{5-3.11}$$

If F and g are determined (as they will be by the conditions

(5-2.21) and (5-2.22) on the surface of the piston) equation (5-3.11) determines the function r.

5-4. The Boundary Conditions on the Piston

In this subsection we discuss the conditions (5-2.21) and (5-2.22). As in subsection (3-1) we write

$$u(s) = C_u + \left(\frac{ac}{A}\right) \operatorname{Im} \int_{\pm t}^{z(s)} \sqrt{\frac{\tau^2 - A^2}{(\tau^2 - a^2)(\tau^2 - c^2)}}\, f(\tau) d\tau$$

$$+ \left(\frac{ac}{A}\right) \int_{\pm\zeta_o}^{\zeta(s)} \sqrt{\frac{\tau^2 - A^2}{(\tau^2 - a^2)(\tau^2 - c^2)}}\, g(\tau) d\tau \qquad (5\text{-}4.1)$$

$$v(s) = C_v + \operatorname{Im} \int_{\pm p}^{z(s)} f(\tau) d\tau + \int_{\pm\zeta_o}^{\zeta(s)} g(\tau) d\tau \qquad (5\text{-}4.2)$$

where s is arc length along the image of the piston in the z-plane (see figure 10) and C_u and C_v are constants. We also have

$$f(z) = \left\{ \frac{2}{\pi} \int_{c}^{\zeta_o} \frac{\xi r(\xi)}{\xi^2 - z^2}\, d\xi \right\} F(z) \qquad (5\text{-}4.3)$$

where the function r has been normalized so that

$$\frac{2}{\pi}\int_{c}^{\zeta_o} \frac{r(\xi)}{\xi}\, d\xi = 1 \qquad (5\text{-}4.4)$$

In the complex integrals along the image of the piston we have that

$$z^2 = O(\varepsilon^2) \qquad (5\text{-}4.5)$$

so that in these integrals we can take

$$f(z) = \left\{\frac{2}{\pi}\int_{c}^{\zeta_o} \frac{r(\xi)}{\xi}\, d\xi\right\} F(z) = F(z) \qquad (5\text{-}4.6)$$

Thus we can rewrite equations (5-4.1) and (5-4.2) in terms of the analytic function F(z), as follows:

$$u(s) = C_u + \left(\frac{ac}{A}\right) \mathrm{Im}\int_{\pm t}^{z(s)} \sqrt{\frac{\tau^2-A^2}{(\tau^2-a^2)(\tau^2-c^2)}}\, F(\tau)d\tau$$

$$+ \left(\frac{ac}{A}\right)\int_{\pm\zeta_o}^{\zeta(s)} \sqrt{\frac{\tau^2-A^2}{(\tau^2-a^2)(\tau^2-c^2)}}\, g(\tau)d\tau \qquad (5\text{-}4.7)$$

$$v(s) = C_V + \mathrm{Im}\int_{\pm p}^{z(s)} F(\tau)d\tau + \int_{\pm\zeta_o}^{\zeta(s)} g(\tau)d\tau \qquad (5\text{-}4.8)$$

We now differentiate, and employ equations (5-2.21) and (5-2.22) so that we have

$$\frac{du}{ds}(s) = \xi_s(s) \qquad (5\text{-}4.9)$$

$$\frac{dv}{ds}(s) = \eta_s(s) \tag{5-4.10}$$

or, using the representations (5-4.7) and (5-4.8) we can write

$$\xi_s(s) = \left(\frac{ac}{A}\right) \mathrm{Re}\left\{\sqrt{\frac{z^2-A^2}{(z^2-a^2)(z^2-c^2)}}\, F(z)\right\} \eta_s(s)$$

$$+ \left(\frac{ac}{A}\right) \mathrm{Im}\left\{\sqrt{\frac{z^2-A^2}{(z^2-a^2)(z^2-c^2)}}\, F(z)\right\} \xi_s(s)$$

$$+ \left(\frac{ac}{A}\right) \sqrt{\frac{\zeta^2-A^2}{(\zeta^2-a^2)(\zeta^2-c^2)}}\, \zeta_s(s) g(\zeta) \tag{5-4.11}$$

$$\eta_s(s) = \mathrm{Re}\left\{F(z)\right\}\eta_s(s) + \mathrm{Im}\left\{F(z)\right\}\xi_s(s) + \xi_s(s)g(\zeta) \tag{5-4.12}$$

where

$$z = z(s) \tag{5-4.13}$$

and

$$\zeta = \zeta(s) \tag{5-4.14}$$

We shall now eliminate $g(\zeta(s))$ from equations (5-4.11) and (5-4.12) to obtain a linear relationship connecting the boundary values of the real and imaginary part of $F(z)$. We first introduce some notation:

$$F(z(s)) = F_r(\xi(s),\eta(s)) + iF_i(\xi(s),\eta(s)) \tag{5-4.15}$$

$$\delta(\tau) = \left(\frac{ac}{A}\right)\sqrt{\frac{\tau^2-A^2}{(\tau^2-a^2)(\tau^2-c^2)}} \qquad (5\text{-}4.16)$$

$$\delta(z(s)) = \delta_r(\xi(s),\eta(s)) + i\delta_i(\xi(s),\eta(s)) \qquad (5\text{-}4.17)$$

$$\Delta(\tau) = \left(\frac{A}{ac}\right)\sqrt{\frac{(\tau^2-a^2)(\tau^2-c^2)}{\tau^2-A^2}} \qquad (5\text{-}4.18)$$

Equations (5-4.11) and (5-4.12) can then be written in the form

$$\xi_s = (\delta_r\eta_s + \delta_i\xi_s)F_r - (\delta_i\eta_s - \delta_r\xi_s)F_i + \Delta^{-1}\zeta_s g \qquad (5\text{-}4.19)$$

$$\eta_s = \eta_s F_r + \xi_s F_i + \zeta_s g \qquad (5\text{-}4.20)$$

Finally, multiplying equation (5-4.19) by Δ and subtracting equation (5-4.20) we obtain the desired linear equation relating the boundary values $F_r(\xi(s),\eta(s))$ and $F_i(\xi(s),\eta(s))$:

$$[(\Delta\delta_r-1)\eta_s + \Delta\delta_i\xi_s]F_r + [(\Delta\delta_r-1)\xi_s - \Delta\delta_i\eta_s]F_i = \Delta\xi_s - \eta_s \qquad (5\text{-}4.21)$$

As we did in the third section, we use the second-order approximation

$$\delta(z(s)) = -i \qquad (5\text{-}4.22)$$

This enables us to write equation (5-4.21) in the form

$$\alpha(s)F_r(s) + \beta(s)F_i(s) = \gamma(s) \tag{5-4.23}$$

where

$$\alpha(s) = \Delta(\zeta(s))\xi_s(s) + \eta_s(s) \tag{5-4.24}$$

$$\beta(s) = \xi_s(s) - \Delta(\zeta(s))\eta_s(s) \tag{5-4.25}$$

$$\gamma(s) = \eta_s(s) - \Delta(\zeta(s))\xi_s(s) \tag{5-4.26}$$

Before discussing the solution of (5-4.23) we devote a subsection to the discussion of the relative parity of the terms in (5-4.21). In the course of this discussion we verify our prior claim regarding the relative parity of the function $g(\zeta)$.

5-5 The Relative Parity of Equation (5-4.21)

In this subsection we will show that the relative parity of equation (5-4.21) is such that we can solve our boundary value problem for $F(z)$ in a quarter plane, and then complete the solution by two applications of the Schwarz reflection principle. In particular, this enables us to construct $F(z)$ so that it will be real for z real or pure imaginary, so long as z lies outside of the image of the piston.

In section four we found the relative parities of $\lambda(\zeta)$ and $\mu(\zeta)$ could be expressed by the following parity diagrams:

$$\lambda(\zeta): \begin{array}{c|c} - & + \\ \hline - & + \end{array} \qquad\qquad \mu(\zeta): \begin{array}{c|c} + & + \\ \hline - & - \end{array}$$

Thus, since

$$\Delta(\zeta) = \frac{\mu(\zeta)}{\lambda(\zeta)} [1 - A^2(\lambda^2(\zeta) + \mu^2(\zeta))]^{-1} \tag{5-5.1}$$

we conclude that the relative parity of $\Delta(\zeta)$ is given by the diagram

$$\Delta(\zeta): \begin{array}{c|c} - & + \\ \hline + & - \end{array}$$

Recalling our definition of arc length (see figure 10) we have immediately that the relative parities of $\xi_s(s)$ and $\eta_s(s)$ are

$$\xi_s(s): \begin{array}{c|c} - & + \\ \hline + & - \end{array} \qquad\qquad \eta_s(s): \begin{array}{c|c} + & + \\ \hline + & + \end{array}$$

Thus, the relative parity of

$$\hat{\gamma}(s) = \Delta(\zeta(s))\xi_s(s) - \eta_s(s) \tag{5-5.2}$$

can be represented by the diagram

$$\hat{\gamma}(s): \quad \begin{array}{c|c} + & + \\ \hline + & + \end{array}$$

A straightforward examination of

$$\delta(z) = \frac{\lambda(z)}{\mu(z)} [1 - A^2(\lambda^2(z) + \mu^2(z))] \tag{5-5.3}$$

discloses that the real and imaginary parts of δ have the relative parities

$$\delta_r: \quad \begin{array}{c|c} - & + \\ \hline + & - \end{array} \qquad\qquad \delta_i: \quad \begin{array}{c|c} + & + \\ \hline + & + \end{array}$$

Thus, the expressions

$$\hat{\alpha} = (\Delta\delta_r - 1)\eta_s + \Delta\delta_i\xi_s \tag{5-5.4}$$

and

$$\hat{\beta} = (\Delta\delta_r - 1)\xi_s - \Delta\delta_i\eta_s \tag{5-5.5}$$

have the relative parities expressed by the following parity diagrams:

$$\hat{\alpha}(s): \quad \begin{array}{c|c} + & + \\ \hline + & + \end{array} \qquad\qquad \hat{\beta}(s): \quad \begin{array}{c|c} - & + \\ \hline + & - \end{array}$$

We can now write equation (5-4.21) in the form

$$\hat{\alpha}(s)F_r(s) + \hat{\beta}(s)F_i(s) = \hat{\gamma}(s) \tag{5-5.6}$$

From this equation, we can deduce that the boundary values of the real and imaginary parts of the analytic function F(z) have the relative parities

$$F_r(s): \quad \begin{array}{c|c} + & + \\ \hline + & + \end{array} \qquad\qquad F_i(s): \quad \begin{array}{c|c} - & + \\ \hline + & - \end{array}$$

Finally, referring to equation (5-4.20), and noting that $\zeta_s(s)$ has the relative parity

$$\zeta_s(s): \quad \begin{array}{c|c} - & + \\ \hline + & - \end{array}$$

we can infer that $g(\zeta)$ has the relative parity stated earlier, i.e.,

$$g(\zeta): \quad \begin{array}{c|c} - & + \\ \hline + & - \end{array}$$

Thus, if we construct an analytic function F(z) whose boundary values satisfy equation (5-4.21) or (5-4.23) in the first quadrant, and extend this function by first reflecting across the positive imaginary axis, and then across the entire real axis, the resulting boundary values will satisfy (5-4.21) or (5-4.23) on the whole image of the piston. This is the strategy we will employ.

5-6. Solution of the Boundary Value Problem: Formal Discussion

In this subsection we discuss the construction of the analytic function F(z). As suggested in the last subsection we shall solve a quarter plane problem. On the piston, we satisfy equation (5-4.23). On the real and imaginary axes we require F(z) to be real. This will make C_u in (5-4.7) zero, and C_v in (5-4.8) of order ε^2. It will also permit us to extend the analytic function F(z) to the whole plane, by applying the reflection principle twice.

We begin by multiplying equation (5-4.23) by a yet-to-be determined real function H(s):

$$H(s)\alpha(s)F_r(s) + H(s)\beta(s)F_i(s) = H(s)\gamma(s) \qquad (5\text{-}6.1)$$

We regard the left hand side of equation (5-6.1) as the boundary values of the imaginary part of the product of two analytic functions, the analytic function F(z) and some other analytic function G(z). From (5-6.1) we deduce that the boundary values of the real part of G(z) are $H(s)\beta(s)$, while those of the imaginary part of G(z) are $H(s)\alpha(s)$. If we write

$$h(z) = \log G(z) \qquad (5\text{-}6.2)$$

we have that

$$\operatorname{Im}\left\{h(z)\right\} = \arg[G(z)] \tag{5-6.3}$$

Thus, the boundary values of $\operatorname{Im}\left\{h(z)\right\}$ on the piston image are

$$\lambda(s) = \tan^{-1}\left(\frac{\alpha(s)}{\beta(s)}\right) \tag{5-6.4}$$

On the real and imaginary axes we take the boundary values of $\operatorname{Im}\left\{h(z)\right\}$ to be zero. We thus consider the quarter plane problem shown in figure 18.

We impose the additional restriction of boundedness at infinity to insure uniqueness. In the next subsection we rigorously discuss the solution of this exterior Dirichlet problem, including a discussion of the singularities which may enter the solution. We complete the determination of the analytic function h(z) by employing the Cauchy-Riemann equations. The function h(z) will contain an arbitrary real additive constant. We shall not carry this constant in our solution since it drops out of the final determination of F(z).

We can now see that

$$H(s) = e^{h_r(s)} \tag{5-6.5}$$

where $h_r(s)$ denotes the boundary values of the real part of h(z). We consider the function

$$\hat{F}(z) = F(z)G(z) \tag{5-6.6}$$

Equation (5-6.1) implies that the boundary values of $\mathrm{Im}\left\{\hat{F}(z)\right\}$ are given by

$$\kappa(s) = e^{h_r(s)}\gamma(s) \tag{5-6.7}$$

Again we take the boundary values of $\mathrm{Im}\left\{\hat{F}(z)\right\}$ to be zero on both the real and imaginary axes, and impose the condition of boundedness at infinity (see figure 19). We postpone the discussion of possible singularities in the solution of this exterior Dirichlet problem until the next subsection.

We construct $\mathrm{Re}\left\{\hat{F}(z)\right\}$ by employing the Cauchy-Riemann equations, noting that this construction introduces an arbitrary real additive constant into the function $\hat{F}(z)$. We will write

$$\hat{F}(z) = \hat{F}_r(\xi,\eta) + \Gamma + i\hat{F}_i(\xi,\eta) \tag{5-6.8}$$

where Γ is the arbitrary constant. We will determine the value of this constant by satisfying the normalization condition (5-3.4).

We now have determined the analytic function F(z) in the first quadrant; it is given by

$$F(z) = \hat{F}(z)e^{-h(z)} \tag{5-6.9}$$

By construction, this function satisfies (5-6.1) on the image of the piston, and is real on both the real and imaginary axes. We extend this function to the whole z-plane (exclusive of the image of the piston) by reflecting across the positive η-axis, and then reflection across the entire ξ-axis. There may be singularities (introduced through discontinuities in the boundary data) at the points $\pm p$, $\pm t$ (see figure 9). In the next subsection we discuss these and other delicate features of the above analysis.

5-7. Solution of the Boundary Value Problem: Rigorous Discussion

In this subsection we shall verify that the procedure outlined in the last subsection can actually be carried out, and examine the possible singularities in the solution. In the boundary value problem for $\text{Im}\left\{h(z)\right\}$ we introduced the function

$$\lambda(s) = \tan^{-1}\left(\frac{\alpha(s)}{\beta(s)}\right) \qquad (5\text{-}7.1)$$

From (5-4.24) and (5-4.25) we have that

$$\frac{\alpha(s)}{\beta(s)} = \frac{\Delta + S}{1-\Delta S} \qquad (5\text{-}7.2)$$

where S is the slope of the image of the piston in the z-plane.

By differentiation we find that the maximum value of Δ (in magnitude) occurs for

$$\zeta_{max} = A^2 \left\{ 1 - \sqrt{1 - 2c^2/A^2} \right\} \tag{5-7.3}$$

Using (5-4.18) it is an easy matter to obtain the bound

$$|\Delta| < 2(A^2 - c^2) \tag{5-7.4}$$

so that Δ is bounded from above and below. Now since S and Δ are nonpositive in the first quadrant, being zero only where the image intersects the ξ-axis, we see that the numerator of (5-7.2) possesses only one zero. Thus, the data on the piston image all lie on a single branch of the arctangent function. We choose the principal branch

$$-\frac{\pi}{2} < \tan^{-1}\theta \leq \frac{\pi}{2} \tag{5-7.5}$$

With this choice, the zero boundary values specified on the coordinate axes are also on the same branch of the arctangent.

There may be a singularity in F(z) at the "front" of the piston, where the boundary values can be discontinuous. Referring to the sketch of the relevant geometry (figure 20), and setting $\Delta = 0$ in this neighborhood of the ξ-axis, we find from (5-7.1) that

$$\lambda_o = \tan^{-1}S_o \tag{5-7.6}$$

So long as

$$-\infty < S_o \leq 0 \tag{5-7.7}$$

we have that

$$-\frac{\pi}{2} < \lambda_o \leq 0 \tag{5-7.8}$$

Thus, for a piston with zero-angle cusps along the undisturbed magnetic axis, the boundary values of the harmonic function $\operatorname{Im}\left\{h(z)\right\}$ will be continuous.

To discuss the general case, we introduce the harmonic function

$$M(S_o)\arg(z - \xi_o) \tag{5-7.9}$$

where

$$M(S_o) = \frac{\tan^{-1}S_o}{\pi + \tan^{-1}S_o} \tag{5-7.10}$$

Using (5-7.8) we see that

$$-1 < M(S_o) \geq 0 \tag{5-7.11}$$

The values of the harmonic function (5-7.9) on the real axis are 0, while on the line segment of slope S_o, the values are $\pi + \tan^{-1}S_o$. Thus, the harmonic function

$$\operatorname{Im}\left\{h(z)\right\} - M(S_o)\arg(z - \xi_o) \tag{5-7.12}$$

assumes zero boundary values in the neighborhood of the "front" of the piston. Thus, locally,

$$\operatorname{Im}\left\{h(z)\right\} \sim M(S_o)\arg(z - \xi_o) \tag{5-7.13}$$

and

$$h(z) \sim \log[(z - \xi_o)^{M(S_o)}] \tag{5-7.14}$$

From (5-7.14) we see that

$$e^{h(z)} \sim (z - \xi_o)^{M(S_o)} \tag{5-7.15}$$

We now examine the boundary value problem for the harmonic function $\operatorname{Im}\left\{\hat{F}(z)\right\}$. On the real axis, the boundary values are 0; on the line segment of slope S_o the boundary values are

$$\eta_s^o |z - \xi_o|^{M(S_o)} \cos\left\{M(S_o)[\pi + \tan^{-1} S_o]\right\} \tag{5-7.16}$$

where η_s^o is a constant. Thus, by employing an argument similar to that used above, we conclude that the local behavior of $\hat{F}(z)$, as far as any singularity is concerned, is

$$\hat{F}(z) \sim (z - \xi_o)^{M(S_o)} \tag{5-7.17}$$

Thus, in the neighborhood of the "front" or "back" of the piston, the analytic function

$$F(z) = \hat{F}(z)e^{-h(z)} \tag{5-7.18}$$

will be regular.

In figure 21 we have sketched the local geometry of the "top" of the piston. Here we have

$$\lambda_o = \tan^{-1} \left(\frac{\delta + S_o}{1-\delta S_o}\right) \tag{5-7.19}$$

where δ is the (negative) limiting value of Δ in this neighborhood, and S_o is the (negative) limiting slope of the image of the piston. Using the addition formula for the tangent, we can write

$$\lambda_o = \tan^{-1}\delta + \tan^{-1}S_o \tag{5-7.20}$$

To discuss the local behavior of the harmonic function $\operatorname{Im}\left\{h(z)\right\}$ near the "top" of the piston, we introduce the harmonic function

$$N(S_o)\arg(z - i\eta_o) \tag{5-7.21}$$

where

$$N(S_o) = \frac{\tan^{-1}\delta + \tan^{-1}S_o}{2\pi + \tan^{-1}S_o} \tag{5-7.22}$$

This function is chosen to have the same local boundary values as $\operatorname{Im}\left\{h(z)\right\}$. Thus, in the neighborhood of the "top" of the piston, the harmonic function

$$\mathrm{Im}\left\{h(z)\right\} - N(S_o)\arg(z - i\eta_o) \qquad (5\text{-}7.23)$$

will assume zero boundary values. Thus, locally,

$$h(z) \sim N(S_o)\log(z - i\eta_o) \qquad (5\text{-}7.24)$$

and

$$e^{h(z)} \sim (z - i\eta_o)^{N(S_o)} \qquad (5\text{-}7.25)$$

To proceed to discuss the boundary value problem for $F(z)$, we must guarantee that

$$-1 < N(S_o) \leq 0 \qquad (5\text{-}7.26)$$

Using the expansion (4-3.15), and recalling that in the neighborhood of the origin the mapping from the $(\frac{x}{t}, \frac{y}{t})$-plane to the z-plane is nearly the identity mapping, we can write

$$\zeta = c[1 + \frac{c^2}{2a^2A^2}\eta^2] \qquad (5\text{-}7.27)$$

Some complicated algebra yields the approximation

$$\delta = -\frac{c^2}{A^2}\eta_o[1 - \frac{c^2}{2}(\frac{1}{a^4} - \frac{1}{A^4})\eta_o^2] \qquad (5\text{-}7.28)$$

so that

$$\tan^{-1}\delta \approx -\frac{c^2}{A^2}\eta_o \qquad (5\text{-}7.29)$$

Since $N(S_o)$ is inherently negative

$$-\tan^{-1}\delta - \tan^{-1}S_o < 2\pi + \tan^{-1}S_o \qquad (5\text{-}7.30)$$

Using (5-7.29) we can write (5-7.30) in the form

$$-\tan^{-1}S_o < \pi - \frac{1}{2}\left(\frac{c^2}{A^2}\right)\eta_o \qquad (5\text{-}7.31)$$

Since η_o is small, (5-7.31) is always satisfied, as is (5-7.26).

Thus, there will be a singularity in the function $\hat{F}(z)$. As before, multiplication by the function $e^{-h(z)}$ will remove this singularity, and $F(z)$ will therefore be regular. Thus, the solution to the boundary value problem will always be regular.

5-8. The Normalization of the Function r

Now that $F(z)$ has been constructed we can use equations (5-4.19) and (5-4.20) to determine $g(\zeta^+)$. Since $F(z)$ contains an arbitrary real constant, so will $g(\zeta^+)$. If we use the approximate values

$$\delta_r = 0 \qquad (5\text{-}8.1)$$

$$\delta_i = -1 \qquad (5\text{-}8.2)$$

equations (5-4.19) and (5-4.20) can be written as

$$\xi_s = -\xi_s F_r + \eta_s F_i + \Delta^{-1}\zeta_s g \tag{5-8.3}$$

$$\eta_s = \eta_s F_r + \xi_s F_i + \zeta_s g \tag{5-8.4}$$

If we multiply (5-8.3) by ξ_s, (5-8.4) by η_s, and subtract we obtain

$$\xi_s^2 - \eta_s^2 = -F_r + (\xi_s \Delta^{-1} - \eta_s)\zeta_s g^+ \tag{5-8.5}$$

where we have used the fact that

$$\xi_s^2 + \eta_s^2 = 1 \tag{5-8.6}$$

We now determine the value of the arbitrary real constant that appears in the solution by satisfying the normalization condition (5-3.4), namely

$$\int_c^{\zeta_o} \frac{r(\xi)}{\xi}\, d\xi = \frac{\pi}{2} \tag{5-8.7}$$

Equation (5-3.11) serves to define $r(\zeta)$ in terms of $F(\zeta)$ and $g(\zeta^+)$. We have

$$r(\zeta) = \frac{g(\zeta^+)}{F(\zeta)} \tag{5-8.8}$$

The variable ζ is real and here lies in the range

$$c \leq \zeta \leq \zeta_o \tag{5-8.9}$$

so that $F(\zeta)$ in the denominator of (5-8.8) is identical with

$F_r(\zeta)$. We shall write

$$F_r(\zeta) = R(\zeta) + \Gamma \tag{5-8.10}$$

where Γ is the undetermined constant. Equation (5-8.5) gives

$$g(\zeta^+) = \frac{R(\zeta) + \Gamma + \xi_s^2 - \eta_s^2}{\zeta_s(\xi_s\Delta^{-1} - \eta_s)} \tag{5-8.11}$$

so that we can write, with the aid of (5-8.8)

$$r(\zeta) = \frac{R(\zeta) + \Gamma + \xi_s^2 - \eta_s^2}{\zeta_s(R(\zeta) + \Gamma)(\xi_s\Delta^{-1} - \eta_s)} \tag{5-8.12}$$

Thus, the constant Γ is to be determined from the condition

$$\int_c^{\zeta_o} \frac{R(\zeta) + \Gamma + \xi_s^2 - \eta_s^2}{\zeta\zeta_s(R(\zeta) + \Gamma)(\xi_s\Delta^{-1} - \eta_s)}\, d\zeta = \frac{\pi}{2} \tag{5-8.13}$$

where

$$s = s(\zeta) \tag{5-8.14}$$

Since the integration in (5-8.13) extends over an interval of length $O(\varepsilon^2)$, and the right hand side is $O(1)$, the integrand must be $O(1/\varepsilon^2)$. This means that $F(z)$ must have nearly a zero in the neighborhood of the point $z = c$. With the determination of the value of the constant Γ, all the desired properties of the solution are fulfilled, and all the functions which appear in the solution are completely determined.

5-9. Extension to Asymmetric Pistons

As we mentioned earlier, symmetry about the $(\frac{x}{t})$-axis is inherent; however, we can allow the piston cross-section to be asymmetric about the $(\frac{y}{t})$-axis. We have already indicated how to write down the hyperbolic wave portion of the solution for such a piston.

If the piston is asymmetric about the $(\frac{y}{t})$-axis we must introduce two functions r_R and r_L, one for the left and one for the right. The analytic function is obtained by solving a half plane problem, and then reflecting across the $(\frac{x}{t})$-axis. The trouble with this procedure is that we are no longer able to make F(z) real on the imaginary axis. Thus, the constant C_u introduced in (5-4.1) will not necessarily vanish.

Since we are matching the derivative of the velocity rather than the velocity itself, the vanishing of C_u is essential to the success of the method. Thus, we can conclude that there will be solutions for an asymmetric piston only if the geometry is such that

$$\operatorname{Im} \int_{\infty}^{t} \sqrt{\frac{\tau^2 - A^2}{(\tau^2 - a^2)(\tau^2 - c^2)}}\, f(\tau) d\tau = 0 \qquad (5\text{-}9.1)$$

This condition is equivalent to a condition on the Green's function of the domain exterior to the image of the piston. It is reasonable to suspect that there will be a class of asymmetric pistons for which conical solutions are possible.

6. The Solution for a Symmetric Piston with a Partial Hyperbolic Wave

In this section we examine the solution for a piston which is rounded at the "top" and "bottom." In this case the characteristics drawn from the cusped portions of the slow wave envelope (which are used to construct the hyperbolic wave) will not cover the whole piston. We would expect free boundary problems in the neighborhood of the "top" and the "bottom" of the piston, while the rest of the piston (covered by the hyperbolic wave) would give rise to the sort of boundary value problem which we discussed in the last section.

In this section we consider a piston cross-section which is slightly perturbed from the circular free boundary obtained earlier. As we add the hyperbolic wave, we seek to determine the perturbations to the free boundary. The free boundary is preserved, to first order, indicating that mixed solutions of this sort are possible. It should be noted that we consider piston expansion velocities so small that the shape of the piston and its image in the z-plane are coincident.

6-1. Formulation of the Problem

In this subsection we denote the unperturbed expansion speed of the piston by r_o, and the perturbation to that speed by $\varepsilon r(\theta)$ (see figure 22). We write

$$\xi(\theta) = (r_o + \varepsilon r(\theta))\cos\theta \qquad (6\text{-}1.1)$$

$$\eta(\theta) = (r_o + \varepsilon r(\theta))\sin\theta \qquad (6\text{-}1.2)$$

We take $r(\theta)$ to be unknown in the angular regions

$$\theta_o < \theta < \pi - \theta_o \qquad (6\text{-}1.3)$$

and

$$\pi + \theta_o < \theta < 2\pi - \theta_o \qquad (6\text{-}1.4)$$

For the rest of the piston we consider $r(\theta)$ as a known function.

In section 3 we obtained the expansion

$$\frac{1}{\lambda} = \pm c\left\{1 + \frac{c^2}{2a^2A^2}\left(\frac{y}{t}\right)^2 + O(r_o^3)\right\} \qquad (6\text{-}1.5)$$

In an entirely analogous manner we obtain the expansion

$$\frac{1}{\mu} = \left(\frac{y}{t}\right)\left\{1 + \frac{1}{c}\left(\frac{x}{t}\right) + \frac{1}{c^2}\left(\frac{x}{t}\right)^2 + O(r_o^3)\right\} \qquad (6\text{-}1.6)$$

Since the point $(\frac{x_o}{t}, \frac{y_o}{t})$ shown in figure 22 is the point of intersection of the last real characteristic from the slow wave envelope, and the piston, we can match the slope of the piston at that point with the slope of the hyperbolic wave. The slope of the hyperbolic wave is

$$\sigma\left(\frac{x}{t}, \frac{y}{t}\right) = \frac{\lambda}{\mu} = -\frac{1}{c}\left(\frac{y}{t}\right)\left\{1 + \frac{1}{c}\left(\frac{x}{t}\right) + O(r_o^2)\right\} \tag{6-1.7}$$

Since the piston is nearly circular, its slope is approximately given by

$$S\left(\frac{x}{t}, \frac{y}{t}\right) = -\left(\frac{x}{t}\right)\Big/\left(\frac{y}{t}\right) \tag{6-1.8}$$

We now set

$$S\left(\frac{x_o}{t}, \frac{y_o}{t}\right) = \sigma\left(\frac{x_o}{t}, \frac{y_o}{t}\right) \tag{6-1.9}$$

The approximate solution of (6-1.9) is

$$\left(\frac{x_o}{t}\right) \approx \frac{r_o^2}{c} \tag{6-1.10}$$

$$\left(\frac{y_o}{t}\right) \approx r_o\left(1 - \frac{r_o^2}{2c^2}\right) \tag{6-1.11}$$

Thus, we see that θ_o is very close to $\frac{\pi}{2}$. On the small intervals at the "top" and "bottom" of the piston we solve a free boundary problem. On the rest of the piston we solve a boundary value problem for both the elliptic solution and the hyperbolic wave. A general existence theorem is very difficult to provide; the analytic function f(z) which results from the various portions of the solution must be shown to be one analytic function in the whole z-plane, excluding the image of the piston.

6-2. The Solution of the Mixed Problem

Using the notation of section 5, we write the function F(z) as

$$F(z) = \frac{r_o^2}{z^2} + \varepsilon \hat{f}(z) \qquad (6\text{-}2.1)$$

Using (6-1.1) and (6-1.2) for the regions where $r(\theta)$ is known we find that the analogue of equation (5-4.23) is the linear relationship

$$(\eta_\theta + \Delta\xi_\theta)F_r(\xi,\eta) + (\xi_\theta - \Delta\eta_\theta)F_i(\xi,\eta) = \eta_\theta - \Delta\xi_\theta \qquad (6\text{-}2.2)$$

The order one terms of this equation constitute an identity; the order ε terms yield

$$\alpha(\theta)\hat{f}_r(\theta) - \beta(\theta)\hat{f}_i(\theta) = \gamma(\theta) \qquad (6\text{-}2.3)$$

where

$$\alpha(\theta) = \cos\theta - \Delta\sin\theta \qquad (6\text{-}2.4)$$

$$\beta(\theta) = \sin\theta + \Delta\cos\theta \qquad (6\text{-}2.5)$$

$$\gamma(\theta) = \frac{2}{r_o}\left[r(\theta)(\cos\theta + \Delta\sin\theta) + r_\theta(\theta)(\sin\theta - \Delta\cos\theta)\right] \qquad (6\text{-}2.6)$$

From section 3 we know that in the two intervals $\theta_o < \theta < \pi - \theta_o$ and $\pi + \theta_o < \theta < 2\pi - \theta_o$ (where the radial perturbation $r(\theta)$ is unknown) we must satisfy the two relations

$$F_r^2 + F_i^2 = 1 \tag{6-2.7}$$

$$S = \frac{F_i}{1 - F_r} \tag{6-2.8}$$

where S is the slope of the image of the piston in the z-plane. Together these two equations imply that

$$\eta_\theta(\theta)F_i(\theta) = \xi_\theta(\theta) + \xi_\theta(\theta)F_r(\theta) \tag{6-2.9}$$

The order one terms again constitute an identity; the first order terms in the parameter ε yield

$$\hat{\alpha}(\theta)\hat{f}_r(\theta) - \hat{\beta}(\theta)\hat{f}_i(\theta) = \hat{\gamma}(\theta) \tag{6-2.10}$$

where

$$\hat{\alpha}(\theta) = - \sin \theta \tag{6-2.11}$$

$$\hat{\beta}(\theta) = \cos \theta \tag{6-2.12}$$

$$\hat{\gamma}(\theta) = \frac{2}{r_o} (r(\theta)\sin \theta - r_\theta(\theta)\cos \theta) \tag{6-2.13}$$

In the two small angular regions where these relations are valid we can set

$$\cos \theta = 0 \tag{6-2.14}$$

$$\sin \theta = \pm 1 \tag{6-2.15}$$

where the plus sign in (6-2.15) refers to the "top" of the piston, and the minus sign refers to the "bottom." Thus, at the "top" of the piston (6-2.10) gives

$$\hat{f}_r(\theta) = \frac{2r(\theta)}{r_o} \tag{6-2.16}$$

while at the "bottom" of the piston this equation gives

$$\hat{f}_r(\theta) = -\frac{2r(\theta)}{r_o} \tag{6-2.17}$$

Since we want F(z) to be real on the real axis, we must have that $\hat{f}_r$ is an even function, upon reflection in the real axis. Thus, we conclude that

$$r(\theta) = 0 \tag{6-2.18}$$

in

$$(\theta_o < \theta < \pi - \theta_o) \cup (\pi + \theta_o < \theta < 2\pi - \theta_o) \tag{6-2.19}$$

Thus, we now know the entire cross-section of the piston, and can employ the methods of the last section to complete the solution. This result indicates that there are solutions to mixed problems of this sort.

7. Solution for a Thin Piston with Finite Expansion Velocity

In this section we obtain the solution for a symmetric piston whose surface is nearly parallel to the $(\frac{x}{t})$-axis. As would be expected, in light of our earlier remarks, the linearization is valid in this case, inasmuch as the perturbations to the flow variables are all $O(\varepsilon)$, where here ε is the maximum of the $(\frac{y}{t})$-component of the piston velocity. We take the maximum $(\frac{x}{t})$-component of the velocity to be $O(1)$ but less than c in magnitude (see figure 23). Finally, we assume that the entire surface of the piston is covered by the hyperbolic wave.

Since the neighborhood of the origin, and the $(\frac{y}{t})$-axis are preserved under the mapping from the $(\frac{x}{t}, \frac{y}{t})$-plane to the z-plane, we can take the image of the piston to be identical with the piston itself.

In the first subsection we discuss the boundary conditions on the surface of the piston. The conditions we obtain are slightly different from those obtained in section one. In particular, we take $\vec{B}\cdot\vec{n} = 0$ on the surface of the piston, so that this surface is flux preserving.

The second subsection discusses the reduction of the problem to the sort discussed in section five. In the last subsection, we treat the piston image as a slit in the z-plane, and write down an explicit formula for the analytic function $F(z)$. We do this by employing the Plemelj integral formulas.

It is worth noting that there appears to be no solution for a thin piston with expansion speed greater than c. In this case the hyperbolic wave will not cover the whole piston, and the remaining portion of the piston does not resemble the circular free boundary obtained in section 3.

7-1. Boundary Conditions on the Piston

On the surface of the piston we require that

$$\vec{u}\cdot\vec{n} = \vec{U}_p\cdot\vec{n} \tag{7-1.1}$$

$$\vec{B}\cdot\vec{n} = \vec{B}_p\cdot\vec{n} \tag{7-1.2}$$

and

$$\vec{B}\cdot\vec{n} = 0 \tag{7-1.3}$$

This last condition insures that the magnetic field is tangential to the surface of the piston, and that the surface is flux preserving. With $B_n = 0$ it is no longer appropriate to prescribe $[E_t] = 0$, [6], since the former condition corresponds to the loss of a characteristic cone.

The fields in the piston are determined by solving the pre-Maxwell equations (without displacement current) with the boundary conditions

$$\vec{B}_p \cdot \vec{n} = 0 \tag{7-1.4}$$

$$\vec{E}_p + \vec{U}_p \times \vec{B}_p = 0 \tag{7-1.5}$$

This is a well-posed problem for the pre-Maxwell equations [3].

If we denote the normal vector to the surface of the piston by

$$\vec{n} = (n_x, n_y) \tag{7-1.6}$$

we see from figure 23 that

$$n_x = 0(\varepsilon) \tag{7-1.7}$$

$$n_y = 0(1) \tag{7-1.8}$$

Using the linearization (1-1.8) we can write (7-1.1) and (7-1.2) in the form

$$u n_x + v n_y = U_1 n_x + U_2 n_y \tag{7-1.9}$$

$$B_x n_x + B_y n_y = - n_x \tag{7-1.10}$$

Since

$$\frac{n_x}{n_y} = - S \tag{7-1.11}$$

where S is the slope of the cross-section of the piston we finally have the boundary conditions

$$v = - SU_1 + U_2 \tag{7-1.12}$$

$$B_y = + S \tag{7-1.13}$$

7-2. Solution of the Boundary Value Problem

We adopt the formalism and notation of section five, and write

$$f(z) = \left\{ \frac{2}{\pi} \int_c^{\zeta_o} \frac{\xi r(\xi)}{\xi^2 - z^2} d\xi \right\} F(z) \tag{7-2.1}$$

In this case, however, we normalize the function r by requiring

$$\int_c^{\zeta_o} \xi r(\xi) d\xi = \frac{\pi}{2} \tag{7-2.2}$$

and on the surface of the piston set

$$f(z) = \frac{F(z)}{c^2 - z^2} \tag{7-2.3}$$

We then satisfy the derivatives of (7-1.12) and (7-1.13), namely,

$$\frac{dv}{ds} = - \xi \frac{d}{ds} \left(\frac{\eta_s}{\xi_s}\right) \tag{7-2.4}$$

$$\frac{dB_y}{ds} = \frac{d}{ds} \left(\frac{\eta_s}{\xi_s}\right) \tag{7-2.5}$$

Straightforward algebra yields the two equations

$$\frac{dv}{ds} = \left[\frac{2\xi\eta\xi_s + (c^2+\eta^2-\xi^2)\eta_s}{(c^2+\eta^2-\xi^2)^2 + 4\xi^2\eta^2} \right] F_r + \left[\frac{(c^2+\eta^2-\xi^2)\xi_s - 2\xi\eta\eta_s}{(c^2+\eta^2-\xi^2)^2 + 4\xi^2\eta^2} \right] F_i + \zeta_s g \tag{7-2.6}$$

$$\frac{dB_y}{ds} = \left[\frac{-2\xi^2\eta\xi_s + \eta(c^2+\eta^2-\xi^2)\xi_s - \xi(c^2+\eta^2-\xi^2)\eta_s - 2\xi\eta^2\eta_s}{(\xi^2+\eta^2)[(c^2+\eta^2-\xi^2)^2 + 4\xi^2\eta^2]} \right] F_r + \left[\frac{-2\xi\eta^2\xi_s - \xi(c^2+\eta^2-\xi^2)\xi_s - \eta(c^2+\eta^2-\xi^2)\eta_s + 2\xi^2\eta\eta_s}{(\xi^2+\eta^2)[(c^2+\eta^2-\xi^2)^2 + 4\xi^2\eta^2]} \right] F_i - \zeta_s\left(\frac{g}{\zeta}\right) \tag{7-2.7}$$

An examination of the relative parity of these two equations shows that the relative parity of the real and imaginary parts of the analytic function F(z) are

$$F_r: \quad \begin{array}{c|c} + & + \\ \hline + & + \end{array} \qquad\qquad F_i: \quad \begin{array}{c|c} - & + \\ \hline + & - \end{array}$$

while the relative parity of the function g is

$$g: \quad \begin{array}{c|c} - & + \\ \hline + & - \end{array}$$

This is just as we found in section five. Thus we can again consider a quarter plane boundary value problem, and extend the resulting function to the whole plane with two applications of the reflection principle.

If we now drop lower order terms in (7-2.6) and (7-2.7) we can write (7-2.4) and (7-2.5) in the form

$$\xi \frac{d}{ds}\left(\frac{\eta_s}{\xi_s}\right) = -\frac{2\xi\eta\xi_s + (c^2-\xi^2)\eta_s}{(c^2-\xi^2)^2} F_r - \frac{\xi_s}{c^2-\xi^2} F_i - \zeta_s g \tag{7-2.8}$$

$$\frac{d}{ds}\left(\frac{\eta_s}{\xi_s}\right) = -\frac{2\xi^2\eta\xi_s - (\eta\xi_s-\xi\eta_s)(c^2-\xi^2)}{(\xi^2+\eta^2)(c^2-\xi^2)^2} F_r - \frac{\xi\xi_s}{(\xi^2+\eta^2)(c^2-\xi^2)} F_i - \frac{\zeta_s g}{\zeta} \tag{7-2.9}$$

We now eliminate g from these two equations to obtain a linear equation connecting the boundary values of the real and imaginary parts of F. We write this relation as

$$\alpha F_r + \beta F_i = \gamma \tag{7-2.10}$$

where

$$\alpha = \frac{[-2\xi\eta\xi_s-(c^2-\xi^2)\eta_s](\xi^2+\eta^2)+\zeta[2\xi^2\eta\xi_s-(\eta\xi_s-\xi\eta_s)(c^2-\xi^2)]}{(\xi^2+\eta^2)(c^2-\xi^2)^2} \tag{7-2.11}$$

$$\approx \frac{-2\xi^3\eta\xi_s-(c^2-\xi^2)\xi^2\eta_s+2\zeta\xi^2\eta\xi_s-\zeta(\eta\xi_s-\xi\eta_s)(c^2-\xi^2)}{(\xi^2+\eta^2)(c^2-\xi^2)} \tag{7-2.12}$$

$$\beta = \xi_s \left[\frac{\xi(\zeta-\xi) - \eta^2}{(\xi^2+\eta^2)(c^2-\xi^2)} \right] \tag{7-2.13}$$

and

$$\gamma = (\xi-\zeta) \frac{d}{ds} \left(\frac{\eta_s}{\xi_s}\right) \tag{7-2.14}$$

We take the piston to be smooth, so that

$$\frac{d}{ds} \left(\frac{\eta_s}{\xi_s}\right) = O(\varepsilon) \tag{7-2.15}$$

Then we can calculate that

$$\alpha = O(\varepsilon) \tag{7-2.16}$$

$$\beta = O(1) \tag{7-2.17}$$

$$\gamma = O(\varepsilon) \tag{7-2.18}$$

One might hope that further simplification could be obtained by setting $\alpha = 0$. However, this reduction sufficiently changes the character of the problem so that no solution is possible.

We have discussed boundary value problems of the sort (7-2.10) in section five. To be able to carry out the solution, we must insure that

$$\tan^{-1} \left(\frac{\alpha(s)}{\beta(s)}\right) \tag{7-2.19}$$

remains on one branch of the arctangent function. We make the approximation $\zeta = c$, and find from (7-2.12) and (7-2.13) that

$$\frac{\alpha}{\beta} = \frac{2\xi^2\eta-(c+\xi)[\xi^2 S+c(\eta-\xi S)]}{(c+\xi)[\xi(c-\xi)-\eta^2]} \qquad (7\text{-}2.20)$$

The denominator of this expression has a zero at

$$\eta = +\sqrt{\xi(c-\xi)} \qquad (7\text{-}2.21)$$

The numerator can be written in the form

$$(c-\xi)[\xi(c+\xi)S - \eta(c+2\xi)] \qquad (7\text{-}2.22)$$

This expression has a zero only for $S \geq 0$. In the first quadrant $S \leq 0$, so that the numerator of (7-2.20) has a zero only when $\eta = 0$ and then only if $S = 0$. Thus, if we take the branch

$$0 \leq \tan^{-1}\theta < \pi \qquad (7\text{-}2.23)$$

the boundary values we specify for the imaginary part of the auxiliary analytic function $h(z)$ all lie on one branch of the arctangent. Thus, the problem reduces to the problem considered in section five, except here, on the piston

$$|f(z)| = O(\varepsilon) \qquad (7\text{-}2.24)$$

since the final boundary data are all $O(\varepsilon)$. Thus in this case the linearized problem appears to be a faithful model of the nonlinear problem.

Our uniqueness theorem no longer holds, since in this case

$$u \neq U_1 \tag{7-2.25}$$

on the piston. The right hand side of equation (1-6.6) is no longer zero; it is a quantity of $O(\varepsilon^2)$.

7-3. An Explicit Determination of the Function F

Referring to the method of section five to solve the sort of boundary value problem posed by (7-2.10), we can write down an explicit formula for the analytic function $F(z)$. We treat the piston image as a slit in the z-plane, and employ the Plemelj formulas.

We write

$$\lambda(\xi) = \tan^{-1}\left(\frac{\alpha(\xi)}{\beta(\xi)}\right) \tag{7-3.1}$$

and note that

$$\lambda(-\xi) = -\lambda(\xi) \tag{7-3.2}$$

We denote the maximum of the $(\frac{x}{t})$-component of the piston velocity by μ, and then write the analytic function $h(z)$ as

$$h(z) = \frac{1}{\pi} \int_{-\mu}^{+\mu} \frac{\lambda(\tau)}{\tau - z} \, d\tau \tag{7-3.3}$$

On the real axis, in the interval $-\mu \leq \xi \leq +\mu$ we have

$$h_r(\xi, 0) = \frac{P}{\pi} \int_{-\mu}^{\mu} \frac{\lambda(\tau)}{\tau - \xi} \, d\tau \tag{7-3.4}$$

The boundary values of $\mathrm{Im}\left\{\hat{F}(z)\right\}$ on the slit from $-\alpha$ to α are

$$\kappa(\xi) = \mathrm{sgn}(\xi)\gamma(\xi)e^{h_r(\xi,0)} \tag{7-3.5}$$

We can then write

$$\hat{F}(z) = \frac{1}{\pi} \int_{-\mu}^{\mu} \frac{\kappa(\tau)}{\tau - z} \, d\tau + \Gamma \tag{7-3.6}$$

where Γ is an arbitrary real constant. The function $F(z)$ is then given by

$$F(z) = \hat{F}(z)e^{-h(z)} \tag{7-3.7}$$

It is easily verified that this function satisfies all the desired conditions. The constant Γ is determined from the normalization condition (7-2.2).

Finally, it is an easy matter to estimate the behavior of F(z) near the points $\pm\mu$ and 0, where it may possess singularities. This is a question of the behavior of Cauchy type integrals near the endpoints of the interval of integration. (To discuss the point 0 we write $\int_{-\mu}^{+\mu} = \int_{-\mu}^{0} + \int_{0}^{+\mu}$, so that zero becomes an endpoint.) To discuss the behavior of

$$C(z) = \frac{1}{2\pi i}\int_a^b \frac{\psi(t)}{t - z}\,dt \qquad (7\text{-}3.8)$$

in the neighborhood of the point a, when $\psi(a) < \infty$, we write

$$C(z) = \frac{1}{2\pi i}\int_a^b \frac{\psi(t) - \psi(a)}{t - z}\,dt + \frac{\psi(a)}{2\pi i}\int_a^b \frac{dt}{t-z} \qquad (7\text{-}3.9)$$

or

$$C(z) = C_o(z) + \frac{\psi(a)}{2\pi i}\log\left(\frac{z - b}{z - a}\right) \qquad (7\text{-}3.10)$$

where

$$C_o = \frac{1}{2\pi i}\int_a^b \frac{\psi(t) - \psi(a)}{t - z}\,dt \qquad (7\text{-}3.11)$$

will be analytic in the neighborhood of z = a. We take a definite branch of the logarithm, which is single-valued in the plane slit from a to b. Then we can write

$$C(z) = C_a(z) + \frac{\psi(a)}{2\pi i}\log\left(\frac{1}{z-a}\right) \qquad (7\text{-}3.12)$$

where $C_a(z)$ is analytic in the neighborhood of $z = a$. A similar argument yields the estimate

$$C(z) = C_b(z) - \frac{\psi(b)}{2\pi i} \log(\frac{1}{z-b}) \tag{7-3.13}$$

where $C_b(z)$ is analytic in the neighborhood of $z = b$.

In the case that

$$\psi(t) = \frac{\psi_R(t)}{(t-a)^\alpha} \tag{7-3.14}$$

where $\psi_R(t)$ is regular at $z = a$, and $0 < \alpha < 1$, an examination of the function

$$\Omega(z) = \int_a^b \frac{dt}{(t-a)^\alpha (t-z)} \tag{7-3.15}$$

shows that $C(z)$ has a singularity of the form

$$\frac{1}{(z-a)^\alpha} \tag{7-3.16}$$

near the point a.

We will now examine the function $h(z)$ near $z = 0$ and $z = +\mu$. If the piston image intersects the positive ξ-axis with a slope S_o, the limiting value of $\lambda(\xi)$, from equations (7-3.1) and (7-2.20) is

$$\lambda_o = \tan^{-1} S_o \tag{7-3.17}$$

Thus, near $z = \mu$, $h(z)$ has the local behavior

$$h(\mu) \sim \log[(z-\mu)^{M(S_o)}] \tag{7-3.18}$$

where

$$M(S_o) = -\left(\frac{\tan^{-1} S_o}{\pi}\right) \tag{7-3.19}$$

By our choice of branch for the arctangent, $0 \leq M(S_o) < 1$. Thus, from (7-3.5) we have

$$\kappa(\mu) \sim (\mu - c)\,\frac{d}{ds}\,(S_o) \tag{7-3.20}$$

so that

$$\hat{F}(\mu) \sim N(S_o)\log(z-\mu) + \Gamma \tag{7-3.21}$$

where

$$N(S_o) = -\frac{\kappa(\mu)}{\pi} \tag{7-3.22}$$

Thus, the function

$$F(z) = \hat{F}(z)e^{-h(z)} \tag{7-3.23}$$

has a singularity of the form

$$(z-\mu)^{-M(S_o)} \tag{7-3.24}$$

which will always be integrable, and thus cause no singularity in the flow.

At the point $z = \nu$, where the piston image intersects the positive η-axis, λ tends to the limiting value

$$\lambda_o = \tan^{-1}(\frac{c}{\nu}) \qquad (7\text{-}3.25)$$

Near $z = \nu$, $h(z)$ has the local behavior

$$h(\nu) \sim \log[(z-\nu)^M] \qquad (7\text{-}3.26)$$

where here

$$M = - \frac{\tan^{-1}(c/\nu)}{\pi} \qquad (7\text{-}3.27)$$

Here $\kappa(\nu)$ is not finite, so we must use our second result (7-3.16) which gives

$$\hat{F}(\nu) \sim (z-\nu)^M \qquad (7\text{-}3.28)$$

Thus, the function

$$F(z) = \hat{F}(z)e^{-h(z)} \qquad (7\text{-}3.29)$$

will be regular at $z = \nu$.

Summing up, the function F will have weak integrable singularities at the "tips" of the piston, but the resulting flow will be regular.

Figure Captions

1. The normal speeds locus
2. Propagation of plane waves
3. The characteristic conoid
4. A t = constant section of the characteristic conoid
5. Tangents to the characteristic conoid
6. Tangents to the characteristic conoid
7. Placement of the piston cross-section
8. The z-plane
9. The image of the piston in the z-plane
10. Definition of arc length on the image of the piston
11. Mapping of the z-plane into the ζ-plane
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17. The z-plane and the image of the piston
18. Quarter-plane problem for $\mathrm{Im}\left\{h(z)\right\}$
19. Quarter-plane problem for $\mathrm{Im}\left\{\hat{F}(z)\right\}$
20. Local boundary value problem for $\mathrm{Im}\left\{h(z)\right\}$
21. Local boundary value problem for $\mathrm{Im}\left\{h(z)\right\}$
22. Piston for the mixed problem
23. Finite-size piston

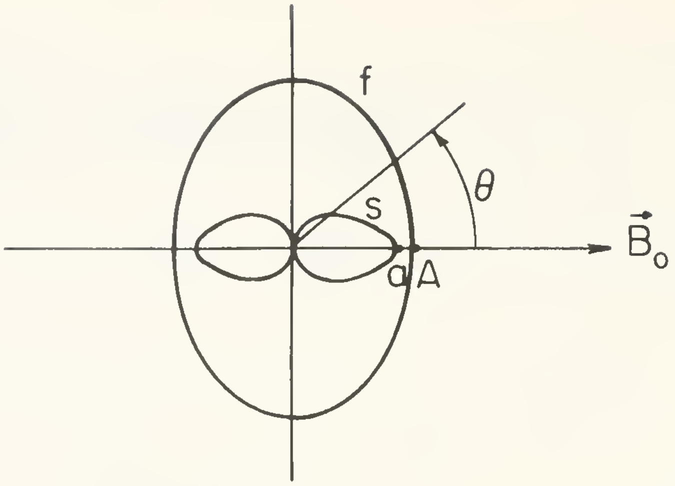


Figure 1

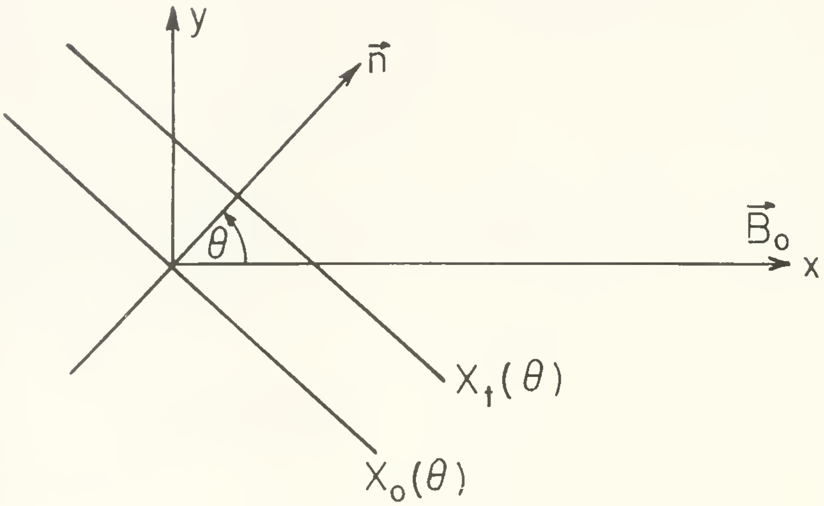


Figure 2

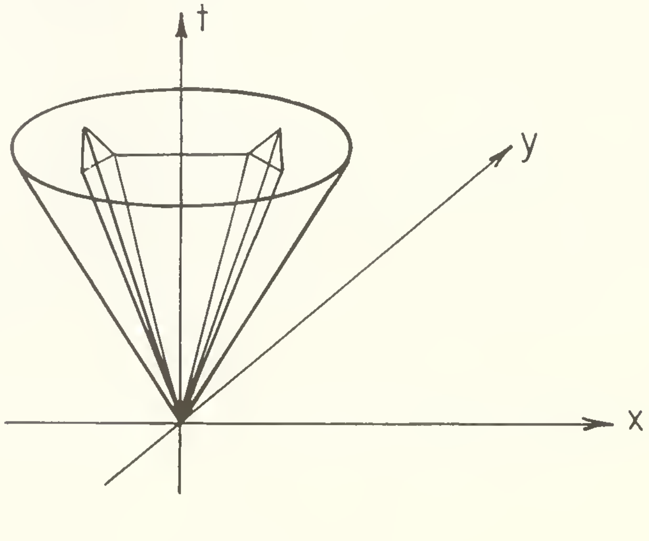


Figure 3

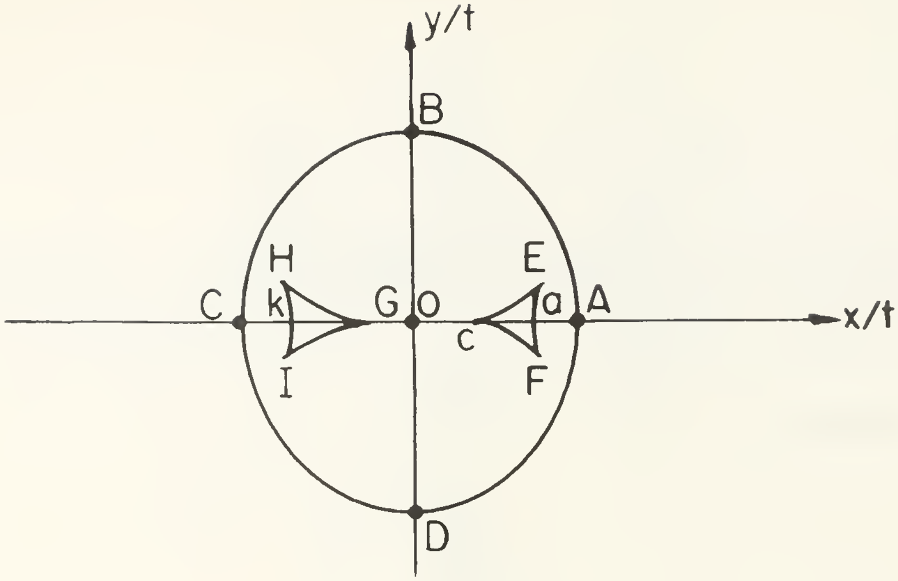


Figure 4

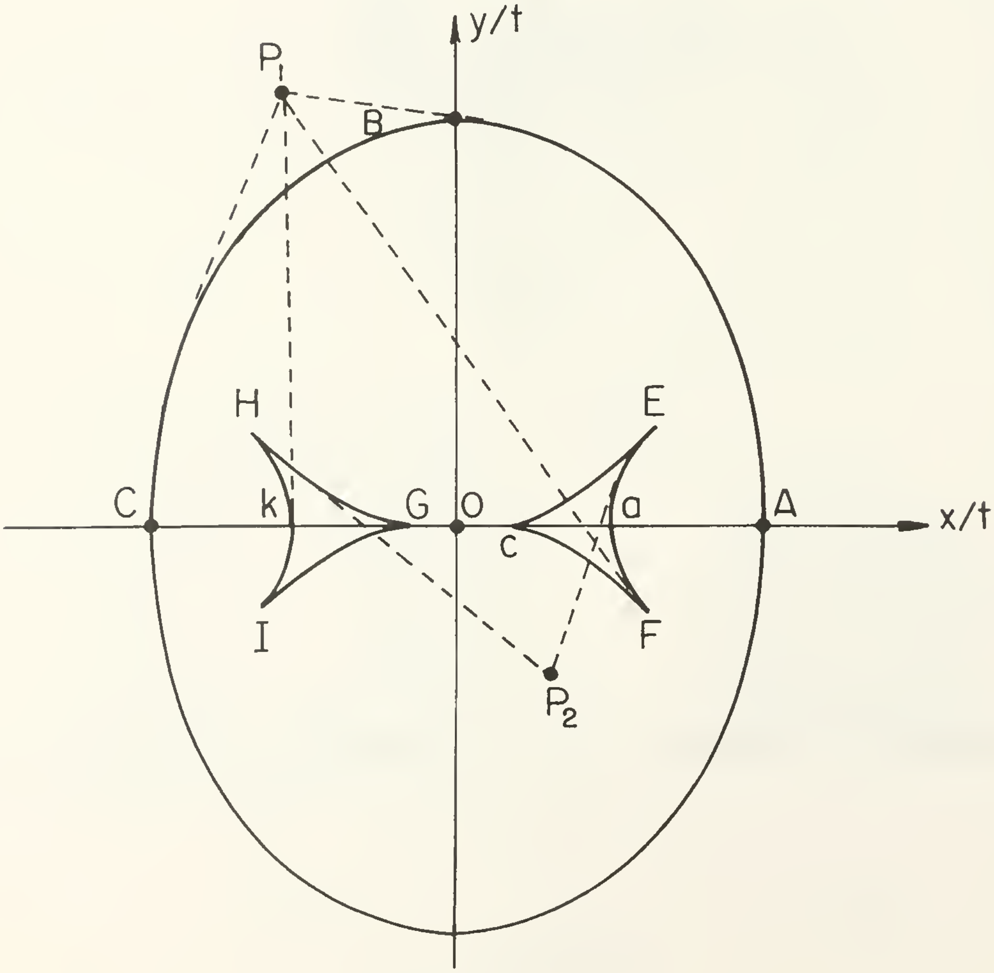


Figure 5

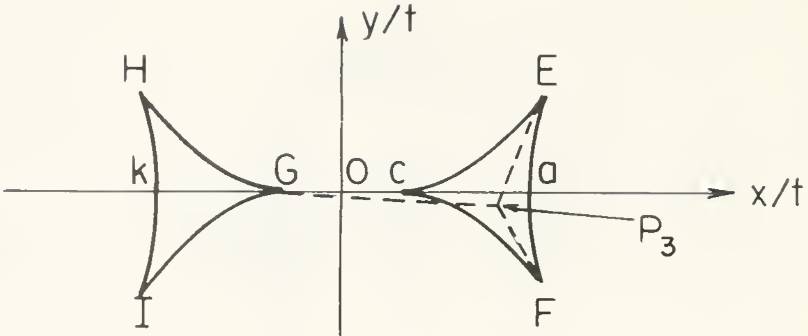


Figure 6

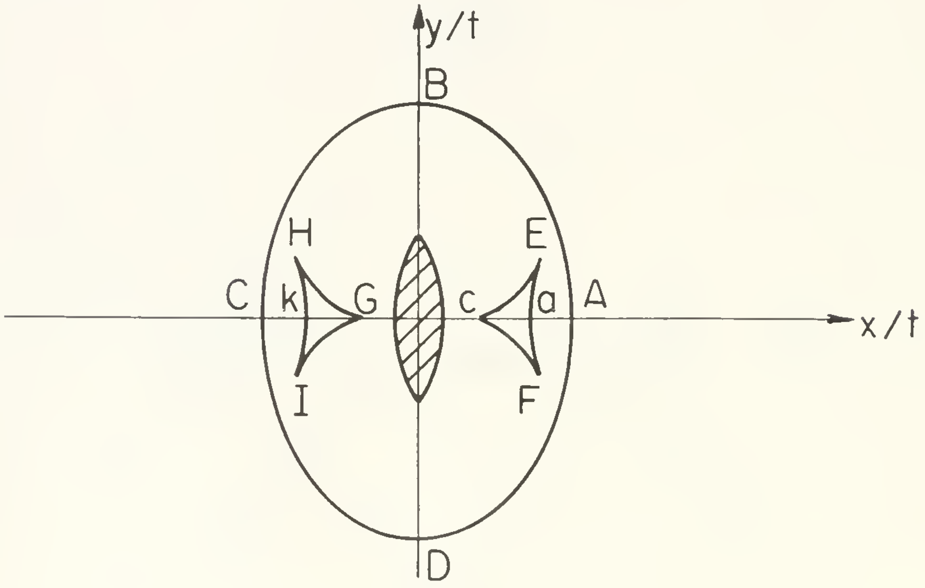


Figure 7

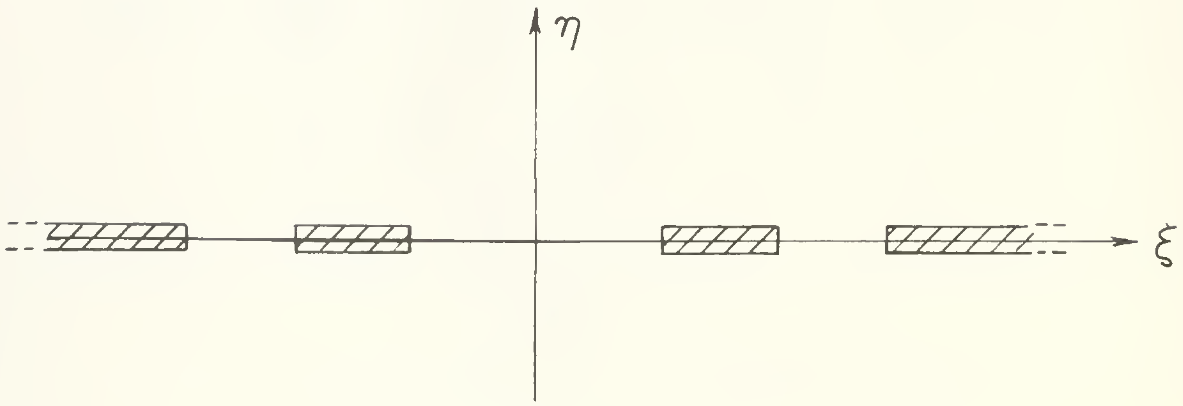


Figure 8

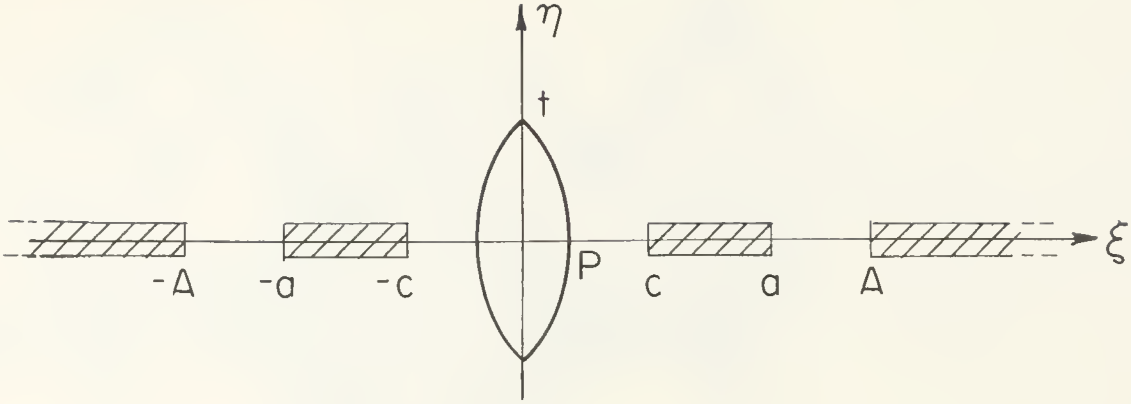


Figure 9

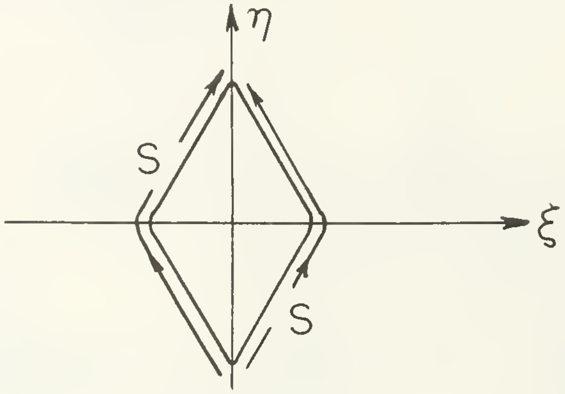
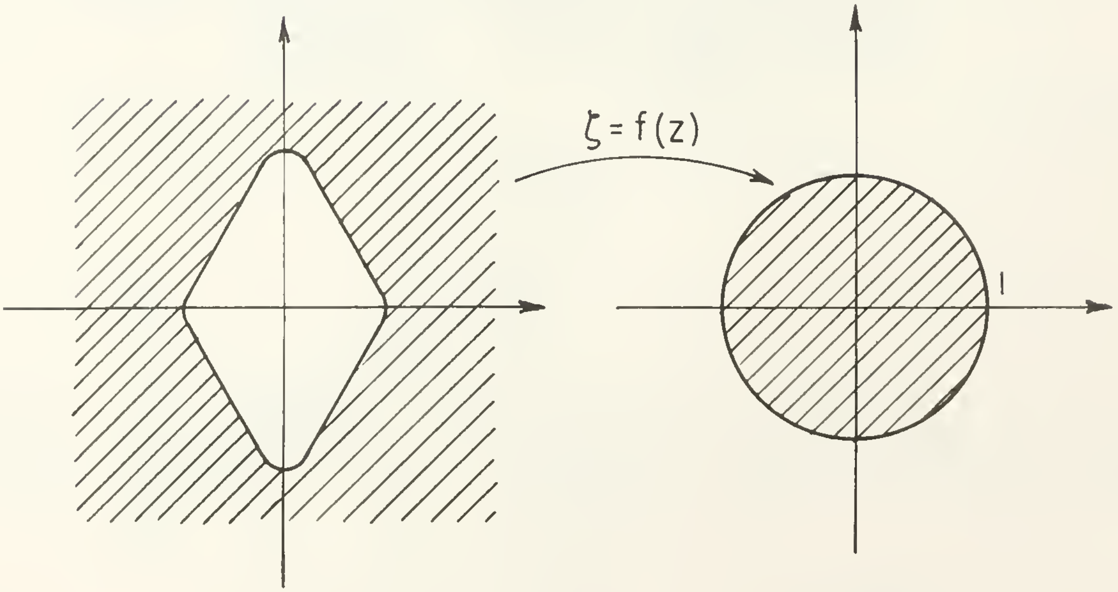


Figure 10



z-plane

ζ-plane

Figure 11

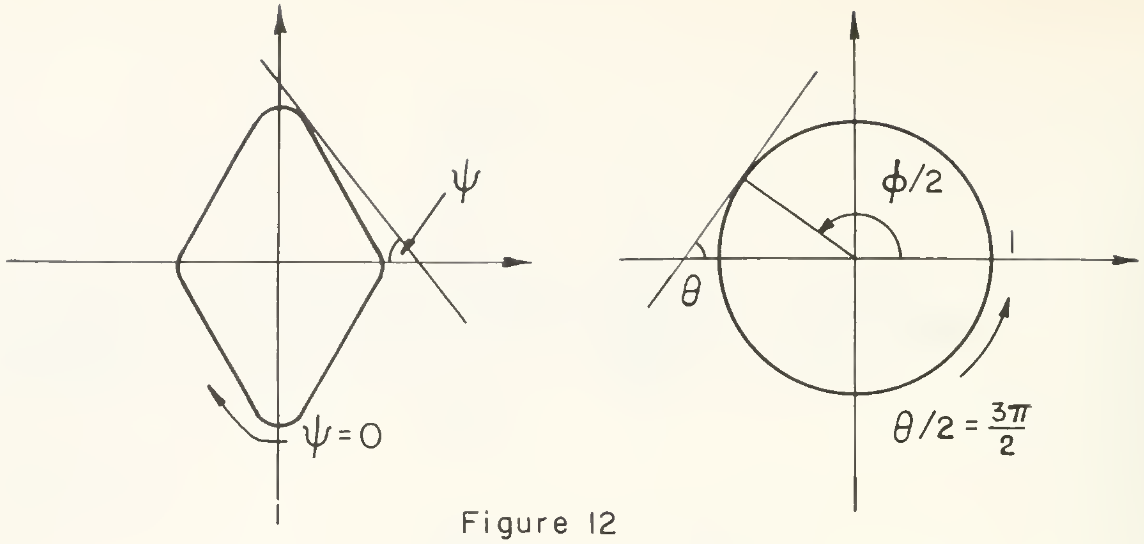


Figure 12

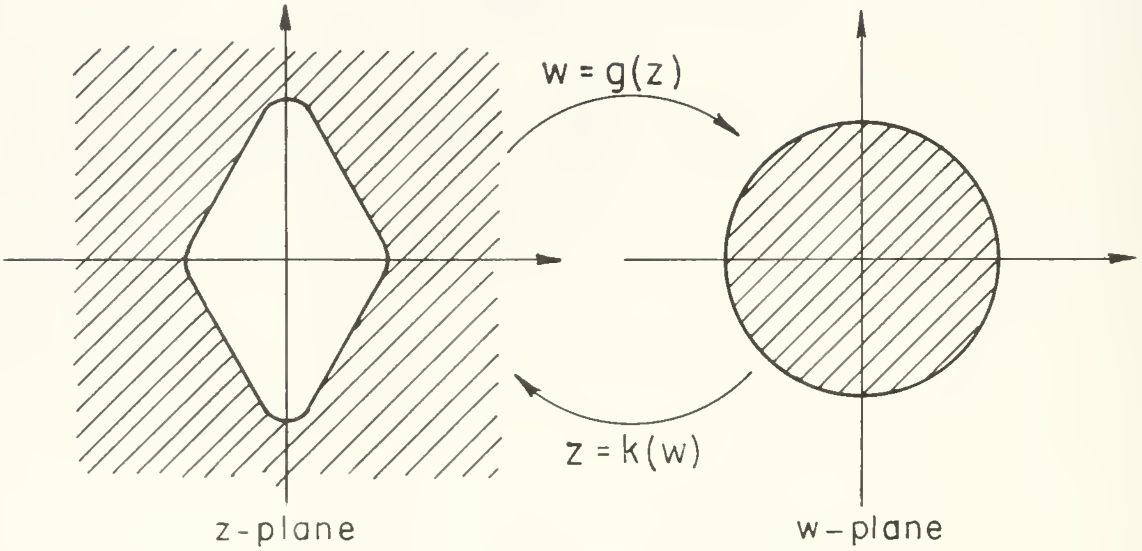
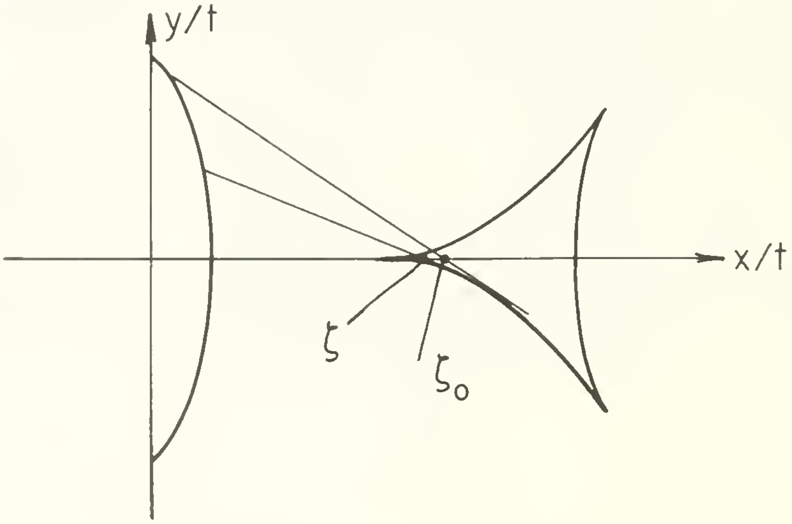


Figure 13



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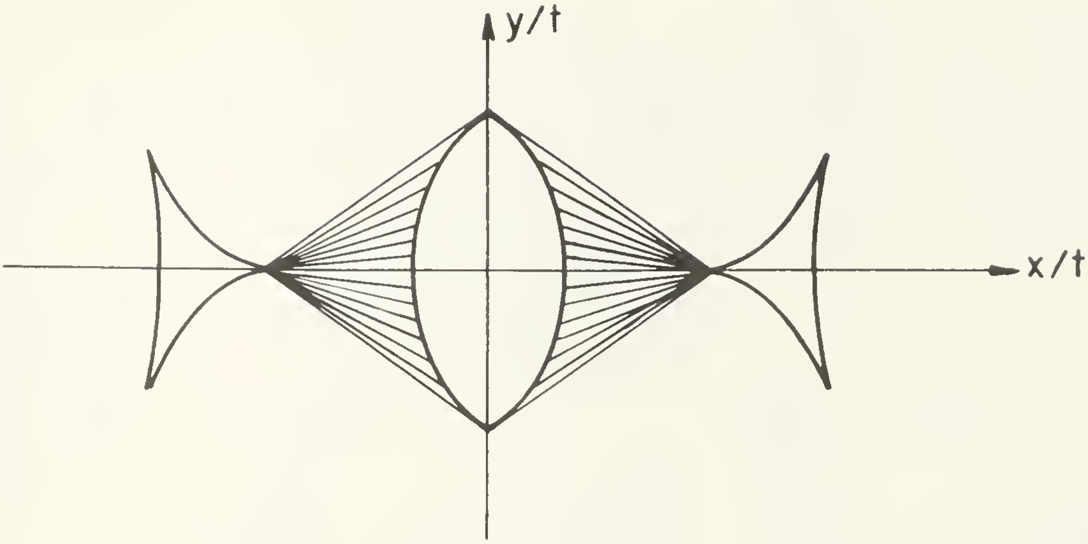

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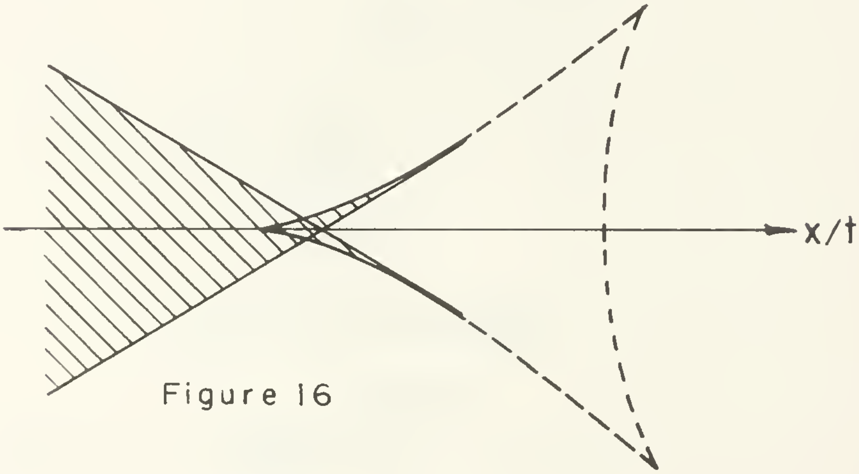

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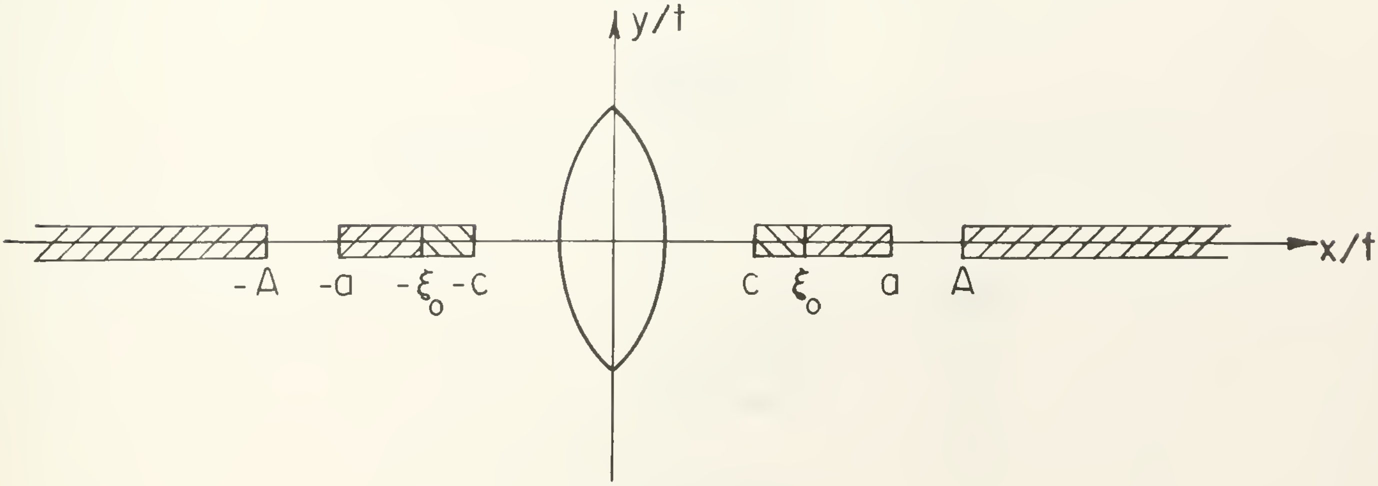

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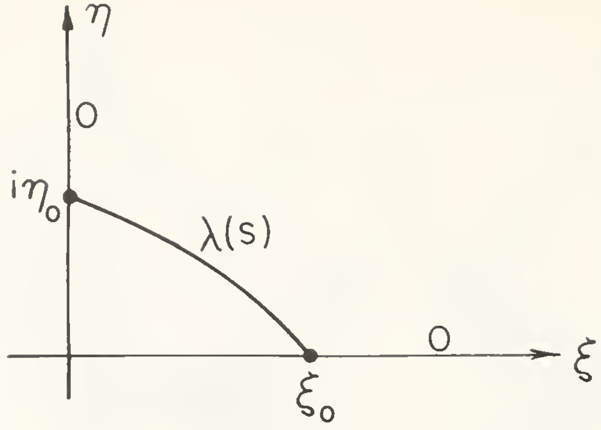

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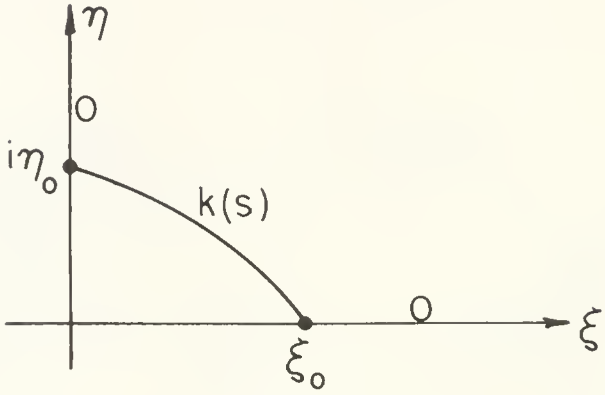

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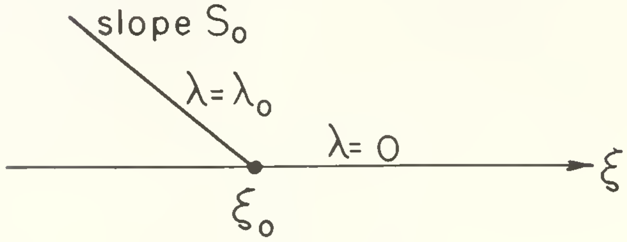

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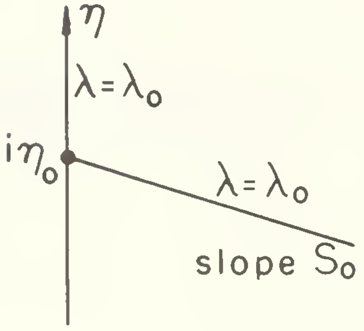

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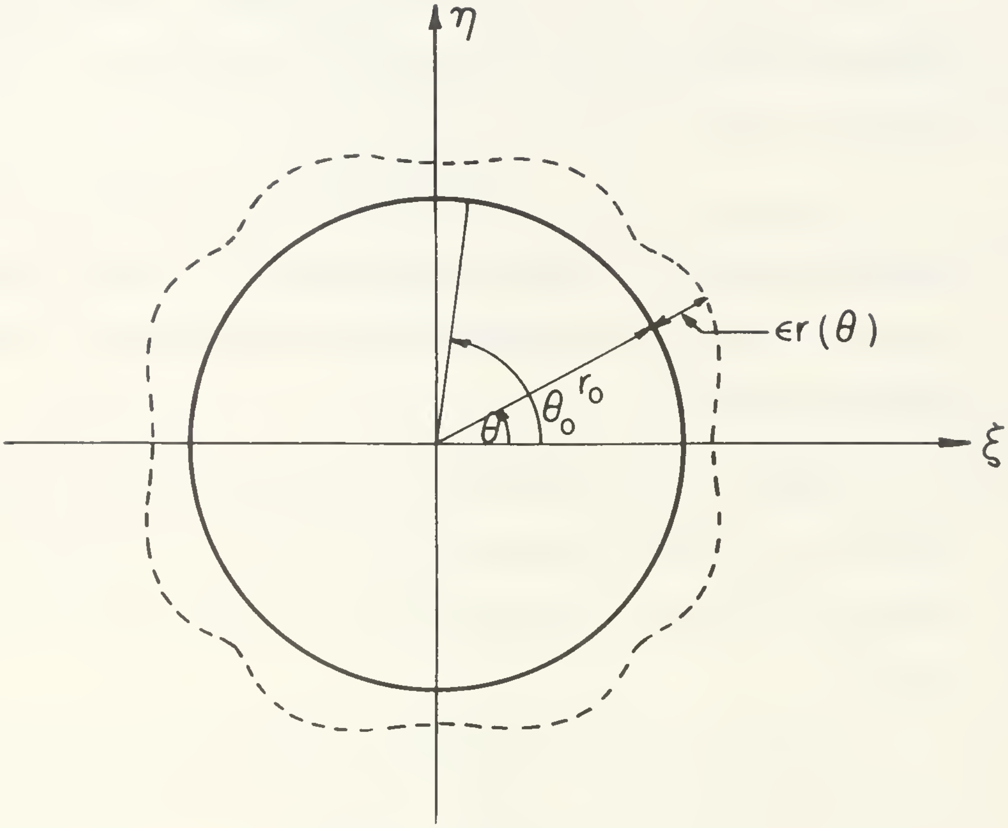

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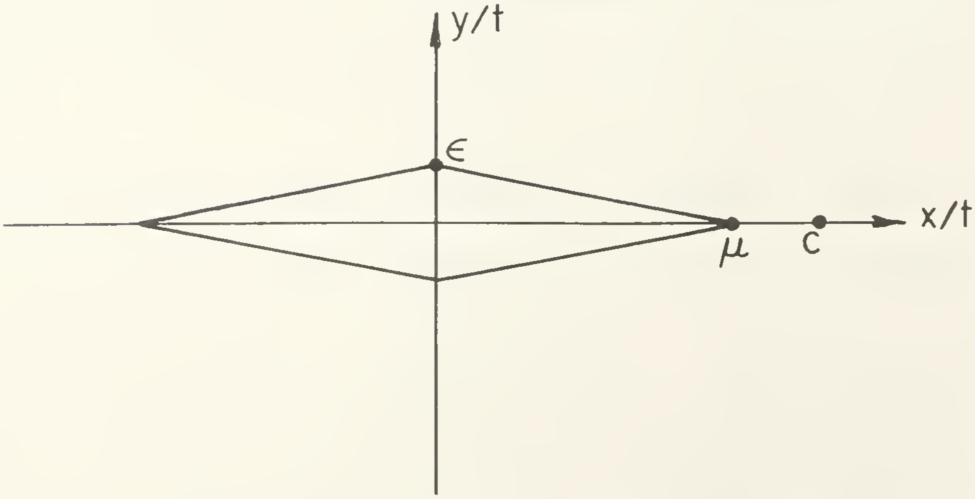

Figure 23

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N.Y.U. Courant Institute of
Mathematical Sciences
251 Mercer St.
New York 12, N. Y.